ISRAEL JOURNAL OF MATHEMATICS, Vol. 74, Nos. 2-3, 1991

ENTROPY OF INNER FUNCTIONS

BY

M. CRAIZER

Departamento de Matemática, C.T.C., P.U.C., RJ,
Rua Marquês de São Vicente 225, Gávea, CEP 22453, Rio de Janeiro, RJ, Brazil

ABSTRACT

In this paper, we show that an inner function f has finite entropy if and only if its derivative f' lies in the Nevanlinna class. We prove also that the entropy of f is given by the average of the logarithm of $|f'|$. The proof is based on the fact that, even f being highly discontinuous on the circle, the action of f^{-n} on Borel subsets is smooth.

Introduction

We shall write $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$ and $\mathbf{S}^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. An *inner function* is a holomorphic map $f: \mathbf{D} \to \mathbf{D}$ such that for a.e. $z \in \mathbf{S}^1$ the radial limit $f^*(z) := \lim_{r \to 1} f(rz)$ belongs to $\mathbf{S}^1$. It is easy to see that f^* preserves Lebesgue measure λ on $\mathbf{S}^1$ if and only if $f(0) = 0$ and in this case f^* is ergodic. Our aim is to characterize the inner functions f for which the entropy $h_\lambda(f^*)$ is finite and to give a formula for calculating it.

Before stating the result, let us recall some preliminary facts. Every holomorphic function $f: \mathbf{D} \to \mathbf{D}$ can be written as

$$f(z) = e^{i\theta} \prod_i \left(\frac{|a_i|}{a_i} \cdot \frac{z - a_i}{1 - \bar{a}_i z} \right) \exp\left(-\int_{\mathbf{S}^1} \frac{t + z}{t - z} \, d\mu(t) \right),$$

$z \in \mathbf{D}$, where μ is a finite positive measure on $\mathbf{S}^1$ and (a_i) is the sequence of zeros of f (it can be empty). This sequence satisfies

$$\sum_i (1 - |a_i|) < \infty.$$

The function f is an inner function if and only if μ is singular with respect to λ.

The function $f^*: \mathbf{S}^1 \to \mathbf{S}^1$ can be very discontinuous. If $z \in \mathbf{S}^1$ is a singular point of f (i.e., z is an accumulation point of the sequence (a_i) or is in the support of μ) then f^* maps every neighborhood of z onto $\mathbf{S}^1$. On the other hand, if z is not a singular point, f extends holomorphically to a neighborhood of z.

The dynamics of a holomorphic map $f:\mathbf{D}\to\mathbf{D}$ is described by the following result due to Denjoy and Wolff [De]:

There exists $p\in\bar{\mathbf{D}}$ such that $\lim_{n\to\infty} f^n(z)=p$ uniformly on compact sets of D. Moreover, if $p\in\mathbf{D}, f(p)=p$ and $|f'(p)|<1$. In particular, a holomorphic map $f:\mathbf{D}\to\mathbf{D}$ has at most one fixed point.

When f is an inner function and p is a fixed point of f, it is easy to prove that the harmonic measure λ_p on $\mathbf{S}^1$ is f^*-invariant. Recall that λ_p can be defined as the unique probability measure such that the integral of a continuous function $\psi:\mathbf{S}^1\to\mathbf{R}$ is given by

$$\int_{\mathbf{S}^1}\psi\, d\lambda_p=\bar{\psi}(p),$$

where $\bar{\psi}$ is the unique extension of ψ which is continuous in $\bar{\mathbf{D}}$ and harmonic in $\mathbf{D}$. It results then from the Poisson formula that

$$\frac{d\lambda_p}{d\lambda}(z)=\operatorname{Re}\frac{z+p}{z-p}.$$

Aaronson [A1] and Neuwirth [N] showed that λ_p is exact, i.e., denoting by $\mathcal{B}(\mathbf{S}^1)$ the Borel σ-algebra of $\mathbf{S}^1$, the σ-algebra $\mathcal{A}:=\bigcap_{n=0}^{\infty}(f^*)^{-n}(\mathcal{B}(\mathbf{S}^1))$ contains only sets of measure zero or one.

On the other hand, it follows from the results of Neuwirth [N] that if f^* has an invariant probability measure μ absolutely continuous with respect to the Lebesgue measure, then f has a fixed point p and $\lambda_p=\mu$. More on the ergodic properties of inner functions can be found in [A2], [Po1] and [Po2].

Our aim is to calculate the entropy of f^* with respect to λ_p when $f(p)=p$. We say that a holomorphic map $g:\mathbf{D}\to\mathbf{C}$ is in the Nevanlinna class (and we denote this by $g\in N$) if

$$\sup_{r<1}\int_{\mathbf{S}^1}\log^+|g(re^{i\theta})|\,d\theta<\infty.$$

In this case, there exists the radial limit $g^*(z)=\lim_{r\to1}g(rz)$ for a.e. $z\in\mathbf{S}^1$ and

$$\int_{\mathbf{S}^1}\big|\log|g^*|\big|\,d\lambda<\infty.$$

THEOREM A. *Let f be an inner function with a fixed point $p\in\mathbf{D}$. Then $h_{\lambda_p}(f^*)<\infty$ if and only if $f'\in N$ and in this case*

$$h_{\lambda_p}(f^*) = \int_{\mathbf{S}^1} \log|(f')^*| \, d\lambda_p.$$

This theorem was conjectured by Fernandez [F] and proved by Martin [M] when the set of singular points of f^* is finite.

Observe that to prove Theorem A we can assume without loss of generality that the fixed point p is $p = 0$ (conjugating f with a Möbius map of $\mathbf{D}$ that maps p to 0). Therefore, from now on, f will be an inner function such that $f(0) = 0$. Theorem A is clearly a consequence of the following two theorems:

THEOREM A.1. *If $h_\lambda(f^*) < \infty$ then $f' \in N$ and*

$$h_\lambda(f^*) \geq \int_{\mathbf{S}^1} \log|(f')^*| \, d\lambda.$$

THEOREM A.2. *If $f' \in N$ then*

$$h_\lambda(f^*) \leq \int_{\mathbf{S}^1} \log|(f')^*| \, d\lambda.$$

To prove Theorem A.1 we shall use a result of Rohlin [R] which says that if T is a measure preserving map of the probability space $(X, \mathcal{A}, \mu)$, where X is a metric space and $\mathcal{A}$ its Borel σ-algebra, and its entropy is finite, then T maps zero measure sets in zero measure sets and X can be partitioned in sets where T is injective. This easily implies that T has a jacobian JT that satisfies

$$\mu(T(A)) = \int_A JT \, d\mu,$$

for every $A \in \mathcal{A}$ such that $T|_A$ is injective. Rohlin also proves that

$$h_\mu(T) \geq \int_{\mathbf{S}^1} \log JT \, d\mu.$$

Theorem A.1 will follow applying these results to f^* and using the partition given above to show, relying on results of Heins [H], that f' has radial limit a.e. Then we shall show that the jacobian of f^* is just $|(f')^*|$ and the Rohlin inequality becomes the inequality in Theorem A.1. Finally, we shall use a result of Ahern and Clark [A–C] to conclude that $f' \in N$.

The proof of Theorem A.2 is subtler. One must keep in mind that Lebesgue measure preserving discontinuous maps of $\mathbf{S}^1$, even being real analytic on an open full measure subset of $\mathbf{S}^1$, can have, due to the discontinuities, an entropy much larger than the average of the logarithm of its derivative. For instance, it follows from Arnoux, Ornstein and Weiss [A–O–W] that there exist interval exchange maps of $\mathbf{S}^1$ with infinite entropy. The proof of Theorem A.2 will be based on the fact that, even f^* being highly discontinuous, the action of $(f^*)^{-n}$ on Borel subsets is smooth. More specifically, we shall show that there exist partitions $\mathcal{P}$ of $\mathbf{S}^1$ into finitely many intervals such that, given $A \in \mathcal{P}$, there exists an open disk D_0 such that $D_0 \cap \mathbf{S}^1 = A$ and a normal family of holomorphic functions $T_j^{(n)}$: $D_0 \to \mathbf{C}$, where $n \in \mathbf{N}$ and $1 \leq j \leq \#\mathcal{P}^{(n)}$ (we are denoting by $\mathcal{P}^{(n)}$ the partition $\mathcal{P} \vee \cdots \vee (f^*)^{-n}(\mathcal{P})$), such that $T_j^{(n)}(A) \subset \mathbf{R}$ and

$$\lambda((f^*)^{-n}(S) \cap B_j) = \int_S T_j^{(n)} \, d\lambda,$$

for all $n \in \mathbf{N}$, $S \subset A$ and $B_j \in \mathcal{P}^{(n)}$.

Theorem A.1 will be proved in the next section and Theorem A.2 in section 2, using certain partitions, that among other properties will satisfy those explained above. The construction of these partitions is the objective of sections 4, 5 and 6.

This paper is a version of my thesis. I wish to thank specially R. Mañé, under whose guidance this work was carried out, and also J. C. Yoccoz, P. Sad, W. de Melo and C. Doering for several helpful conversations and corrections of the first draft of the paper.

1. Proof of Theorem A.1

We start with a general result. Let X be a separable metric space and μ be a probability on $\mathcal{B}(X)$. Let $F: (X,\mathcal{B}(X),\mu) \to (X,\mathcal{B}(X),\mu)$ be an endomorphism with $h_\mu(F) < \infty$. Rohlin proved that F is countable to one (see definition in [Pa]) and consequently satisfies the following properties:

(1.a) F is positively measurable, i.e., if $A \in \mathcal{B}(X)$ then $F(A) \in \mathcal{B}(X)$ (mod 0).

(1.b) F is positively non-singular, i.e., if $A \in \mathcal{B}(X)$ and $\mu(A) = 0$ then $\mu(F(A)) = 0$.

(1.c) There exist disjoint Borel sets $A_1, A_2, \ldots$ such that $\mu(\bigcup A_i) = 1$ and $F|_{A_i}$ is injective, $\forall i \in \mathbf{N}$.

Using these properties, it is easy to prove the existence and uniqueness of a function JF, called the jacobian of F, such that

$$\mu(F(A)) = \int_A JF \, d\mu,$$

when $A \in \mathcal{B}(X)$ and $F|_A$ is injective.

Rohlin also proved that

$$h_\mu(F) \geq \int_X \log JF \, d\mu.$$

Let us introduce the angular derivative. We say that an inner function f has an angular derivative at $x \in \mathbf{S}^1$ if $f^*(x)$ exists and has modulus 1 and if $(f')^*(x) = \lim_{r \to 1} f'(rx)$ exists. If f fails to have an angular derivative we shall write $|(f')^*(x)| = \infty$. Note that this does not imply that $|f'(rx)| \to \infty$ as $r \to 1$.

If f has an angular derivative at x, then for every $\alpha > 1$

$$\lim_{\substack{z \to x \\ z \in \Gamma_\alpha(x)}} \frac{f(z) - f^*(x)}{z - x} = (f')^*(x),$$

where

$$\Gamma_\alpha(x) = \left\{ z \in \mathbf{D} \,\middle|\, \frac{|x - z|}{1 - |z|} < \alpha \right\}$$

is the Stoltz angle. More details about the angular derivative can be found in [Ca,¶298,299].

We shall need the following two theorems, proved in [A–C].

Theorem 1.1. *If f is an inner function given by*

$$f(z) = e^{i\theta} \prod_i \left(\frac{|a_i|}{a_i} \cdot \frac{z - a_i}{1 - \bar{a}_i z} \right) \exp\left(- \int_{\mathbf{S}^1} \frac{t + z}{t - z} \, d\mu(t) \right)$$

then, for all $x \in \mathbf{S}^1$,

$$|(f')^*(x)| = \sum_i \frac{1 - |a_i|^2}{|x - a_i|^2} + 2 \int_{\mathbf{S}^1} |x - t|^{-2} \, d\mu(t).$$

Theorem 1.2. *If f is an inner function such that* $\log^+ |(f')^*| \in \mathcal{L}^1$ *then* $f' \in N$.

Proposition 1.3. *Let* $f: \mathbf{D} \to \mathbf{D}$ *be an inner function with* $f(0) = 0$. *Suppose that* $f^*: \mathbf{S}^1 \to \mathbf{S}^1$ *satisfies the properties* (1.a), (1.b) *and* (1.c) *with* $F = f^*$. *Then the jacobian of* f^* *is equal to* $|(f')^*|$.

The proof of this proposition is given below. Another proof can be found in [H].

It is now easy to prove Theorem A.1. We know that $h_\lambda(f^*) < \infty$ implies that f^* is countable to one [see Pa, ch. 10]. By Proposition 1.3 and the considerations above, it follows that

$$h_\lambda(f^*) \geq \int_{S^1} \log|(f')^*|\, d\lambda.$$

It remains to prove that $f' \in N$. But since $f(0) = 0$, using Theorem 1.1, we have that $|(f')^*(x)| \geq 1$, $\forall x \in S^1$. Hence $\log|(f')^*| = \log^+|(f')^*|$ and therefore $\log^+|(f')^*| \in \mathcal{L}^1$. Thus $f' \in N$, by Theorem 1.2.

The rest of this section is devoted to the proof of Proposition 1.3.

Heins showed in [H] that if $A \in \mathcal{B}(S^1)$ is such that $f^*|_A$ is injective, then f has angular derivatives at a.e. $x \in A$. Therefore if f^* satisfies (1.a), (1.b) and (1.c) then f has angular derivatives a.e.

Definition 1.4. We say that $f^*: S^1 \to S^1$ is *almost uniformly differentiable* if for every $\epsilon_0 > 0$ there exists $E \subset S^1$ with $\lambda(E) > 1 - \epsilon_0$ and such that $f^*|_E$ is uniformly differentiable, i.e.,

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that if } x \in E,\ x + h \in E,\ |h| < \delta \text{ then}$$

$$|f^*(x+h) - f^*(x) - g(x)\cdot h| < \epsilon \cdot |h|,$$

where $g: S^1 \to \mathbf{C}$ is a function. This function is called the derivative of f^* and is denoted by $(f^*)'$.

Lemma 1.5. *Suppose that f^* has angular derivatives a.e. Then f^* is almost uniformly differentiable and*

$$(f^*)' = (f')^*.$$

Proof. Given $\epsilon_0 > 0$, there exists $E \in \mathcal{B}(S^1)$ with $\lambda(E) > 1 - \epsilon_0$ and satisfying the following:

Given $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in E$, $x + h \in E$ and $|h| < \delta$ then

$$|(f')^*(x+h) - (f')^*(x)| \leq \epsilon,$$

$$|f(z) - f^*(x) - (f')^*(x)\cdot(z - x)| < \epsilon \cdot |z - x|$$

and

$$|f(z) - f^*(x+h) - (f')^*(x+h)\cdot(z - x - h)| < \epsilon \cdot |z - x - h|,$$

where z is the point of intersection of the line that passes through x making an angle $\pi/4$ with the radius joining x to 0 with the line that passes through $x+h$ making an angle $-\pi/4$ with the radius joining $x+h$ to 0 (see Fig. 1). Observe that z satisfies

$$|z-x| = \frac{\sqrt{2}}{2}\cdot|h| \quad \text{and} \quad |z-x-h| = \frac{\sqrt{2}}{2}\cdot|h|.$$

Since

$$\begin{aligned} &|f^*(x+h)-f^*(x)-(f')^*(x)\cdot h| \\ &= |f^*(x+h)-f(z)+f(z)-f^*(x)-(f')^*(x) \\ &\quad \cdot(z-x)+(f')^*(x)(z-x-h)| \\ &\le |f(z)-f^*(x+h)-(f')^*(x) \\ &\quad \cdot(z-x-h)| + |f(z)-f^*(x)-(f')^*(x)\cdot(z-x)|, \end{aligned}$$

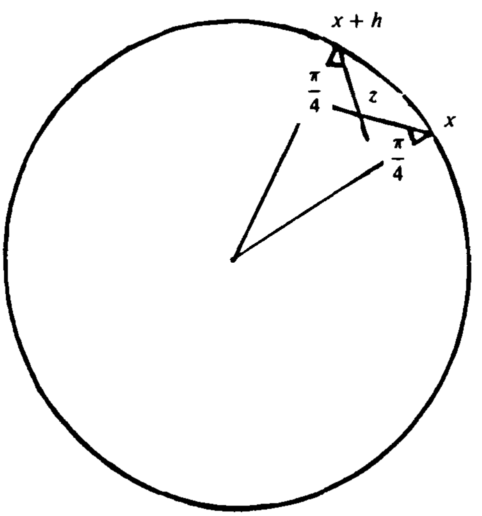

Fig. 1.

it follows that

$$\begin{aligned}
&|f^*(x+h) - f^*(x) - (f')^*(x)\cdot h| \\
&\quad \le |f(z) - f^*(x+h) - (f')^*(x+h)\cdot(z-x-h)| \\
&\qquad + |[(f')^*(x+h) - (f')^*(x)]\cdot(z-x-h)| \\
&\qquad + |f(z) - f^*(x) - (f')^*(x)\cdot(z-x)| \\
&\quad \le \frac{3\sqrt{2}}{2}\cdot\epsilon\cdot|h|.
\end{aligned}$$

Hence f^* is uniformly differentiable in E with derivative $(f')^*$. Therefore f^* is almost uniformly differentiable with derivative $(f')^*$.

Let us now prove Proposition 1.3. Since f^* satisfies (1.a), (1.b) and (1.c), there exist sets $A_1, A_2, \ldots$ with $\lambda(\cup A_i) = 1$ and such that $f^*|_{A_i}$ is injective. Define μ_i as the measure on A_i given by

$$\mu_i(S) = \lambda(f^*(S)),$$

$\forall S \subset A_i$. Then, by the definition of the jacobian,

$$\frac{d\mu_i}{d\lambda} = J(f^*).$$

Proposition 1.3 is a consequence of the following lemma:

LEMMA 1.6. *Suppose that f^* is almost uniformly differentiable. Let $A \in \mathbf{S}^1$ be such that $f^*|_A$ is injective and define the measure μ on A by*

$$\mu(S) = \lambda(f^*(S)),$$

$\forall S \subset A$. Then

$$\frac{d\mu}{d\lambda} = |(f^*)'|.$$

PROOF. Let $E \subset A$ be a Borel set with $\lambda(E) > \lambda(A) - \epsilon_0$ and such that $f^*|_E$ is uniformly differentiable.

CLAIM. $\mu(S) \leq \int_S (|(f^*)'| + \epsilon)\, d\lambda$, where $\epsilon > 0$ and $S \subset E$ are arbitrary.

Indeed, take $\delta > 0$ such that $|h| < \delta$ then

$$|f^*(x+h) - f^*(x) - (f^*)'(x)\cdot h| < \epsilon \cdot |h|,$$

for all $x \in E$ and $x + h \in E$. Let $\{I_1, \ldots, I_r\}$ be a covering of S by open disjoint intervals of length less than δ and such that

$$\frac{\lambda(I_j)}{\lambda(I_j \cap S)} \leq 1 + \gamma,$$

$\forall 1 \leq j \leq r$, where $\gamma > 0$ is arbitrary. Then if $x_j \in S \cap I_j$, we have

$$\lambda(f^*(S \cap I_j)) \leq (|(f^*)'(x_j)| + \epsilon)\cdot \lambda(I_j)$$

and therefore

$$\lambda(f^*(S)) \leq \sum_{j=1}^{r} (|(f^*)'(x_j)| + \epsilon)\cdot \lambda(I_j).$$

This implies that

$$\lambda(f^*(S)) \leq \sum_{j=1}^{r} \frac{\lambda(I_j)}{\lambda(S \cap I_j)} \int_{S \cap I_j} (|(f^*)'| + \epsilon)\, d\lambda$$
$$\leq (1+\gamma) \int_S (|(f^*)'| + \epsilon)\, d\lambda.$$

Since γ is arbitrary, this proves the claim.

Since ϵ and ϵ_0 are also arbitrary, it follows from the claim that

$$\frac{d\mu}{d\lambda} \leq |(f^*)'|.$$

At points where $(f^*)'$ vanishes, the equality holds trivially. At the other points, apply the same reasoning to the inverse of $f^*|_A$, that we denote by g, and use Lemma 1.7, that is proved below, to conclude that

$$\frac{d\mu}{d\lambda} \geq \frac{1}{|g'|} = |(f^*)'|.$$

This proves Lemma 1.6.

LEMMA 1.7. *Suppose that $f^*|_A$ is injective and has a derivative satisfying $(f^*)'(x) \neq 0$, $\forall x \in A$. Let $g: f^*(A) \to A$ be the inverse of $f^*|_A$. Then g is almost uniformly differentiable and its derivative is $g'(x) = 1/(f^*)'(x)$.*

PROOF. Let $E \subset A$ be a compact set with $\lambda(E) \geq \lambda(A) - \epsilon_0$ and such that $f^*|_E$ is uniformly differentiable with $|(f^*)'(x)| > c > 0$, $\forall x \in E$, for some $c > 0$. Fix $y = f^*(x)$. Suppose that $y + k \in f^*(E)$, i.e., $y + k = f^*(x + h)$, for some h satisfying $x + h \in E$. Then

$$\left| g(y+k) - g(y) - \frac{k}{(f^*)'(x)} \right| = \left| h - \frac{(f^*(x+h) - f^*(x))}{(f^*)'(x)} \right|$$

$$\leq \frac{|f^*(x+h) - f^*(x) - (f^*)'(x) \cdot h|}{c}.$$

If we take $\epsilon < c/2$, and $|h| < \delta(\epsilon)$, then

$$|k| \geq \left(\frac{c}{2}\right) \cdot |h|.$$

It follows that for $|k| < (c/2) \cdot \delta$ we have

$$\left| g(y+k) - g(y) - \frac{k}{(f^*)'(x)} \right| \leq \frac{1}{c} \cdot \epsilon \cdot |h|$$

$$\leq 2 \cdot \epsilon \cdot \frac{|k|}{c^2}.$$

This proves that $g|_{f^*(E)}$ is uniformly differentiable with derivative $g'(x) = 1/(f^*)'(x)$. Hence g is almost uniformly differentiable and $g'(x) = 1/(f^*)'(x)$.

2. Transition functions, jacobians and the proof of Theorem A.2

In this section we reduce the proof of Theorem A.2 to Theorem 2.2 below.

Let $F: (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$ be an endomorphism of a probability space $(X, \mathcal{A}, \mu)$. Given $A, B \in \mathcal{A}$ define the transition function, $T_{AB}F: A \to \mathbf{R}$, by the property

$$\int_S T_{AB}F \, d\mu = \mu(F^{-1}(S) \cap B),$$

$\forall S \subset A$. By the theorem of Radon-Nykodim $T_{AB}F$ exists and is unique.

Let $\mathcal{P}$ be a partition of $(X,\mathcal{A},\mu)$. We denote by $\mathcal{P}^{(n)}(x)$ the atom of $\mathcal{P}^{(n)} := \mathcal{P} \vee \cdots \vee F^{-n}\mathcal{P}$ that contains x. Let $x \in \mathbf{S}^1$ be such that $T_{\mathcal{P}(F^n x)\mathcal{P}^{(n)}(x)} F^n(F^n x) \neq 0$. Define the nth jacobian of F with respect to $\mathcal{P}$ at x by

$$J_{\mathcal{P}}^{(n)} F(x) = [T_{\mathcal{P}(F^n x)\mathcal{P}^{(n)}(x)} F^n(F^n x)]^{-1}.$$

LEMMA 2.1. *Suppose that* $T_{\mathcal{P}(F^n x)\mathcal{P}^{(n)}(x)} F^n(u) \neq 0$, *for a.e.* $u \in \mathcal{P}(F^n x)$. *Then*

$$\int_{F^{-n}(S)\cap\mathcal{P}^{(n)}(x)} J_{\mathcal{P}}^{(n)} F \, d\mu = \mu(S),$$

$\forall S \subset \mathcal{P}(F^n x)$.

PROOF. Write $\mathcal{H} = \mathcal{A} \cap \mathcal{P}(F^n x)$. On the space $(\mathcal{P}(F^n x),\mathcal{H})$ we have the measures $\eta_1(S) = \mu(S)$ and $\eta_2(S) = \mu(F^{-n}S \cap \mathcal{P}^{(n)}(x))$.

By definition,

$$\frac{d\eta_2}{d\eta_1} = T_{\mathcal{P}(F^n x)\mathcal{P}^{(n)}(x)} F^n.$$

Write $\mathcal{F} = F^{-n}\mathcal{H} \cap \mathcal{P}^{(n)}(x)$. On the space $(\mathcal{P}^{(n)}(x),\mathcal{F})$ we have the measures $\xi_1(A) = \mu(A)$ and $\xi_2(A) = \mu(S)$, where $A = F^{-n}S \cap \mathcal{P}^{(n)}(x)$. We can prove, using the hypothesis of the lemma, that ξ_2 is well defined.

Since $(F^n)^*\eta_1 = \xi_2$ and $(F^n)^*\eta_2 = \xi_1$, we have

$$\frac{d\xi_1}{d\xi_2} = \frac{d\eta_2}{d\eta_1} \circ F^n$$

and therefore

$$\frac{d\xi_2}{d\xi_1} = \left(\frac{d\eta_2}{d\eta_1} \circ F^n\right)^{-1}.$$

We conclude that

$$\int_{F^{-n}(S)\cap\mathcal{P}^{(n)}(x)} J_{\mathcal{P}}^{(n)} F \, d\mu = \mu(S),$$

proving Lemma 2.1.

Recall that if $H: I \to \mathbf{R}$ is a function defined on the interval I, the oscillation of H is defined by

$$\operatorname{osc} H = \sup_{x,y \in I} |H(x) - H(y)|.$$

THEOREM 2.2. *Given $\epsilon > 0$, there exists a partition $\mathcal{P} = \mathcal{P}_\epsilon = \{I_1, \ldots, I_p\}$ of $\mathbf{S}^1$ into intervals satisfying the following properties*:

(P1) *Write $\mathcal{P}^{(n)} = \{B_1, \ldots, B_s\}$. Then $T_{I_i B_j}(f^*)^n$ is real analytic, $\forall 1 \leq i \leq p$, $1 \leq j \leq s$.*

(P2) *Let $B_1, \ldots, B_r$ be the atoms of $\mathcal{P}^{(n)}$ such that $(f^*)^n(B_j) = I_i$, where $2 \leq i \leq p-1$ and $B_{r+1}, \ldots, B_s$ be the atoms of $\mathcal{P}^{(n)}$ such that $(f^*)^n(B_j) = I_i$, where $i = 1$ or $i = p$.*

Then there exists $A = A(\epsilon) > 0$ independent of n such that:

$$\sum_{j=1}^{r} \operatorname{osc}(T_{I_i B_j}(f^*)^n) \leq A.$$

(P3) *If $i = 1$ or $i = p$, then*

$$|I_i| \leq \epsilon.$$

(P4) $p \cdot \epsilon \leq 1$.

Moreover, if (ϵ_l) is a sequence decreasing to zero, we have

(P5) $\mathcal{P}_{\epsilon_1} \leq \mathcal{P}_{\epsilon_2} \leq \cdots$

and

(P6) $\bigvee_{\substack{l \geq 1 \\ n \geq 0}} (f^*)^{-n}\mathcal{P}_{\epsilon_l} = \mathcal{B}(\mathbf{S}^1)$.

The proof of this theorem is the aim of sections 3, 4, 5 and 6. Let us see how Theorem A.2 follows from Theorem 2.2.

In the calculations that follow, we shall write f instead of f^*.

We write $B_j = \mathcal{P}^{(n)}(x)$ and $I_i = \mathcal{P}(f^n x)$. It follows from property (P1) that if $\lambda(B_j) > 0$ then $T_{I_i B_j} f^n(u) \neq 0$, for a.e. $u \in I_i$. It results then from Lemma 2.1 that

$$\lambda(I_i) = \int_{B_j} J_{\mathcal{P}}^{(n)} f(y)\, d\lambda(y).$$

Then

$$(*) \qquad \frac{\lambda(I_i)}{\lambda(B_j)} = J_{\mathcal{P}}^{(n)} f(x) \left[1 + \frac{1}{\lambda(B_j)} \int_{B_j} \frac{J_{\mathcal{P}}^{(n)} f(y) - J_{\mathcal{P}}^{(n)} f(x)}{J_{\mathcal{P}}^{(n)} f(x)}\, d\lambda(y)\right].$$

But

$$\frac{J_{\mathcal{P}}^{(n)} f(y) - J_{\mathcal{P}}^{(n)} f(x)}{J_{\mathcal{P}}^{(n)} f(y) J_{\mathcal{P}}^{(n)} f(x)} = T_{I_i B_j} f^n(f^n y) - T_{I_i B_j} f^n(f^n x)$$

and therefore

$$\frac{J_{\wp}^{(n)}f(y) - J_{\wp}^{(n)}f(x)}{J_{\wp}^{(n)}f(x)} \leq J_{\wp}^{(n)}f(y)\operatorname{osc}(T_{I_iB_j}f^n).$$

Replacing in (*), we have

$$\frac{\lambda(I_i)}{\lambda(B_j)} \leq J_{\wp}^{(n)}f(x)\left[1 + \frac{1}{\lambda(B_j)}\operatorname{osc}(T_{I_iB_j}f^n)\int_{B_j} J_{\wp}^{(n)}f(y)\,d\lambda(y)\right].$$

Using Lemma 2.1 and the inequality $e^x \geq 1 + x$, which holds for $\forall x \in \mathbf{R}$, we obtain

$$\frac{\lambda(I_i)}{\lambda(B_j)} \leq J_{\wp}^{(n)}f(x)\exp\left\{\frac{\lambda(I_i)}{\lambda(B_j)}\operatorname{osc}(T_{I_iB_j}f^n)\right\}.$$

Write $c = \inf_{1\leq i\leq p}\lambda(I_i)$. Then

$$\frac{c}{\lambda(B_j)} \leq J_{\wp}^{(n)}f(x)\exp\left\{\frac{\operatorname{osc}(T_{I_iB_j}f^n)}{\lambda(B_j)}\right\}.$$

Take logarithms in the above equation and integrate over B_j. It results that

$$(\log c - \log\lambda(B_j))\lambda(B_j) \leq \int_{B_j}\log J_{\wp}^{(n)}f\,d\lambda + \operatorname{osc}(T_{I_iB_j}f^n).$$

Summing in j from 1 to r,

$$(**)\qquad \begin{aligned} &\log c\sum_{j=1}^{r}\lambda(B_j) - \sum_{j=1}^{r}\lambda(B_j)\log\lambda(B_j) \\ &\leq \int_{\bigcup_{j=1}^{r}B_j}\log J_{\wp}^{(n)}f\,d\lambda + \sum_{j=1}^{r}\operatorname{osc}(T_{I_iB_j}f^n). \end{aligned}$$

From property (P3) it results that $\sum_{j=1}^{r}\lambda(B_j) \geq 1 - 2\epsilon$. Then, from the inequality

$$-\sum_{j=1}^{t}a_j\log a_j \leq \sum_{j=1}^{t}a_j\left(\log t - \log\sum_{j=1}^{t}a_j\right),$$

which is valid for every $0 \leq a_j \leq 1$, $1 \leq j \leq t$, we have

$$-\sum_{j=r+1}^{s}\lambda(B_j)\log\lambda(B_j) \leq 2\epsilon[\log(s - (r+1)) - \log 2\epsilon]$$

and since $s - (r+1) \le 2p^n$, we conclude that

$$-\sum_{j=r+1}^{s} \lambda(B_j)\log\lambda(B_j) \le -2\epsilon\log 2\epsilon + 2n\epsilon\log 2p.$$

Replacing these inequalities in (**) and using property (P2) we obtain

$$\log c(1-2\epsilon) + H_\lambda(\mathcal{P}^{(n)}) - 2\epsilon\log\frac{1}{2\epsilon} - 2n\epsilon\log 2p \le \int_{\mathbf{S}^1}\log J_{\mathcal{P}}^{(n)} f\, d\lambda + A.$$

Divide now the two members of the inequality by n and make n go to infinity. It results that

$$h_\lambda(f,\mathcal{P}) - 2\epsilon\log 2p \le \limsup_{n\to\infty}\frac{1}{n}\int_{\mathbf{S}^1}\log J_{\mathcal{P}}^{(n)} f\, d\lambda.$$

It follows from property (P4) that the second term of the first member of the inequality above tends to zero, when ϵ decreases to zero. Hence

$$\lim_{\epsilon\to 0} h_\lambda(f,\mathcal{P}_\epsilon) \le \limsup_{n\to\infty}\frac{1}{n}\int_{\mathbf{S}^1}\log J_{\mathcal{P}}^{(n)} f\, d\lambda.$$

The limit of the first member exists by property (P5). Besides, it follows from (P6) that this limit is exactly $h_\lambda(f)$. Thus

$$(***) \qquad h_\lambda(f) \le \limsup_{n\to\infty}\frac{1}{n}\int_{\mathbf{S}^1}\log J_{\mathcal{P}}^{(n)} f\, d\lambda.$$

PROPOSITION 2.3. *Let f be an inner function with finite angular derivative a.e. Then*

(1) $(f^n)^* = (f^*)^n$ *and*

(2) f^n *has finite angular derivative a.e., with*

$$|[(f^n)']^*(x)| = \prod_{j=0}^{n-1} |(f')^*(f^j x)|$$

for a.e. $x \in \mathbf{S}^1$.

PROOF. Suppose that f has finite angular derivative $(f')^*(y)$ at the point y. Then the image of any curve which is orthogonal to $\mathbf{S}^1$ at y is orthogonal to S^1 at $f^*(y)$. This is sufficient to prove (1). Moreover, if $0 < r < 1$

$$|(f^n)'(rx)| = \prod_{j=0}^{n-1} |f'(f^j(rx))|.$$

Making r tend to 1 and using the observation above we conclude that

$$|[(f^n)']^*(x)| = \prod_{j=0}^{n-1} |(f')^*(f^j(x))|$$

for a.e. $x \in \mathbf{S}^1$. This proves (2).

PROPOSITION 2.4. *Let f be an inner function, $f(0) = 0$, and $\mathcal{P}$ be a finite partition of $\mathbf{S}^1$. Suppose that $f' \in N$. Then*

$$J_{\mathcal{P}}^{(n)} f^*(x) \leq |[(f^n)']^*(x)|$$

for a.e. $x \in \mathbf{S}^1$.

PROOF. Since $f' \in N$, f has finite angular derivative at a.e. $x \in \mathbf{S}^1$. It follows then from Lemma 2.3 that the same holds for f^n and that $(f^n)^* = (f^*)^n$. Heins proved in [H] that in this case $(f^*)^n$ satisfies the properties (1.a), (1.b) and (1.c) from section 1. We have then the jacobian of $(f^*)^n$, denoted by $J[(f^*)^n]$, as in section 1. And it follows from Proposition 1.3 that $J[(f^*)^n] = |[(f^n)']^*|$.

Let $A, B \in \mathcal{B}(\mathbf{S}^1)$ be such that $(f^*)^n(B) = A$. Then it follows from the definition of the jacobian that

$$\int_B J[(f^*)^n]\, d\lambda \geq \lambda(A),$$

with equality holding if and only if $(f^*)^n|_B$ is injective. Suppose now that $A \in \mathcal{P}((f^*)^n(x))$ and $B \in \mathcal{P}^{(n)}(x)$. It follows then from Lemma 2.1 that

$$\int_B J_{\mathcal{P}}^{(n)} f^*\, d\lambda = \lambda(A).$$

Hence

$$J_{\mathcal{P}}^{(n)} f^*(x) \leq J[(f^*)^n](x) = |[(f^n)']^*(x)|,$$

for a.e. $x \in \mathbf{S}^1$.

Using Lemma 2.3 and Proposition 2.4 we conclude that

$$\frac{1}{n}\int_{\mathbf{S}^1} \log J_{\mathcal{P}}^{(n)} f\, d\lambda \leq \int_{\mathbf{S}^1} \log|(f')^*|\, d\lambda.$$

This, together with (***), proves Theorem A.2.

3. Distortion lemma

Let $g : \mathbf{C} \to \mathbf{C}$ be a finite Blaschke product with $g(0) = 0$. Let $U \subset \mathbf{C}$ be an open set conformally equivalent to $\mathbf{D}$, not containing 0 and symmetric with respect to $\mathbf{S}^1$. Let V be a union of connected components of $g^{-1}(U)$.

Define $G_1 : U \to \mathbf{R}$ by

$$G_1(z) = \sum_{\substack{g(w)=z \\ w \in V}} \log|w|.$$

It is clear that G_1 is well defined and harmonic on $U \setminus V(g)$, where $V(g)$ is the set of critical values of g. Moreover, G_1 is bounded. We can therefore extend it to a harmonic function on U.

Let $G : U \to \mathbf{C}$ be a holomorphic map such that $\operatorname{Re} G = G_1$. G is unique except for an additive constant.

Lemma 3.1. (1) $G'(z) \neq 0$, $\forall z \in U \cap \mathbf{S}^1$.

(2) *Write* $U \cap \mathbf{S}^1 = (a, b)$. *Then*

$$G(b) - G(a) = i \int_a^b |G'| \, d\lambda.$$

(3) $\int_a^b |G'| \, d\lambda = \lambda(V \cap \mathbf{S}^1)$.

Proof. (1) Consider a point $z_0 \in U \cap \mathbf{S}^1$. Since G is a holomorphic map, G is equivalent to the map $z \to (z - z_0)^k$ in a neighborhood of z_0. But G takes $\mathbf{D} \cap U$ into $\operatorname{Re}(z) < 0$ and $\mathbf{D}^c \cap U$ into $\operatorname{Re}(z) > 0$. Therefore $k = 1$ and hence $G'(z_0) \neq 0$.

(2) Indeed, G takes $U \cap \mathbf{S}^1 = (a, b)$ injectively onto an interval of the imaginary axis. Hence

$$G(b) - G(a) = i \int_a^b |G'| \, d\lambda.$$

(3) Suppose that $z \in U \setminus V(g)$. Then

$$G'(z) = \sum_{\substack{g(w)=z \\ w \in V}} \frac{1}{w} w'.$$

Hence, for every $z \in U \cap \mathbf{S}^1$,

$$G'(z) = \frac{1}{z} \sum_{\substack{g(w)=z \\ w \in V}} \frac{1}{|g'(w)|}$$

and therefore

$$|G'(z)| = \sum_{\substack{g(w)=z \\ w\in V}} \frac{1}{|g'(w)|}.$$

From this, (3) follows easily.

When we want to emphasize the dependence of G with respect to the Blaschke product g and the union of connected components V of $g^{-1}(U)$, we denote G by $G_{g,V}$.

We recall that a family $\mathcal{F}$ of holomorphic functions in Ω is said to be normal if every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of Ω.

Proposition 3.2. *Write $\mathcal{G} = \{G_{g,V} \mid g$ is a finite Blaschke product and V is a union of connected components of $g^{-1}(U)\}$. Then $\mathcal{G}$ is a normal family.*

Proof. Write $E(l_1, l_2) = \{z \in \mathbf{C} \mid z = iy,\ l_1 < y < l_2\}^c$. It follows then from 3.1(2) that $G_{g,V}$ avoids the set $E(-iG(a), -iG(b))$ and from 3.1(3) that $-iG(b) + iG(a) \leq 1$.

Thus, given a function in $\mathcal{G}$, it omits the segment $E(1,2)$ or else the segment $E(4,5)$. Therefore, for each sequence in $\mathcal{G}$, there exists a subsequence that omits a whole segment of the imaginary axis. Hence, by Montel's theorem, this subsequence has a convergent subsequence, proving the proposition.

Write $\mathbf{D}_r = \{z \in \mathbf{C} \mid |z| < r\}$.

Definition 3.3. Let $\gamma > 1$ be a real number. Let U and U_γ be open sets conformally equivalent to $\mathbf{D}$ such that $U \subset U_\gamma$ and let $\psi : U_\gamma \to \mathbf{C}$ be a Riemann map of U_γ. We say that U_γ is a *γ-extension* of U if $\psi(U) \subset \mathbf{D}_{1/\gamma}$.

Consider a γ-extension U_γ of U.

Let $\bar{G} : U_\gamma \to \mathbf{C}$ be a holomorphic map whose real part is given by

$$\operatorname{Re} \bar{G}(z) = \sum_{\substack{g(w)=z \\ w\in V_\gamma}} \log|w|,$$

where V_γ is the union of the connected components of $g^{-1}(U_\gamma)$ containing V.

Proposition 3.4. *Suppose that $\bar{G}|_U = G$. Then there exists a constant $A = A(\gamma, U_\gamma)$ such that*

$$|G''(z)| \le A\lambda(V_\gamma \cap \mathbf{S}^1),$$

$\forall x \in U$.

Proof. Consider the family

$$\mathcal{H} = \left\{ H = \frac{i\bar{G}}{\bar{G}(b) - \bar{G}(a)} \right\},$$

where $H = H_{g, V_\gamma}$ varies with the Blaschke product g and the union of connected components V_γ of $g^{-1}(U_\gamma)$. We can prove, as in Proposition 3.2, that $\mathcal{H}$ is a normal family (see Fig. 2).

Hence there exists $B = B(\gamma, U_\gamma)$ such that

$$|H(z)| \le B,$$

$\forall z \in U$, $\forall H \in \mathcal{H}$.

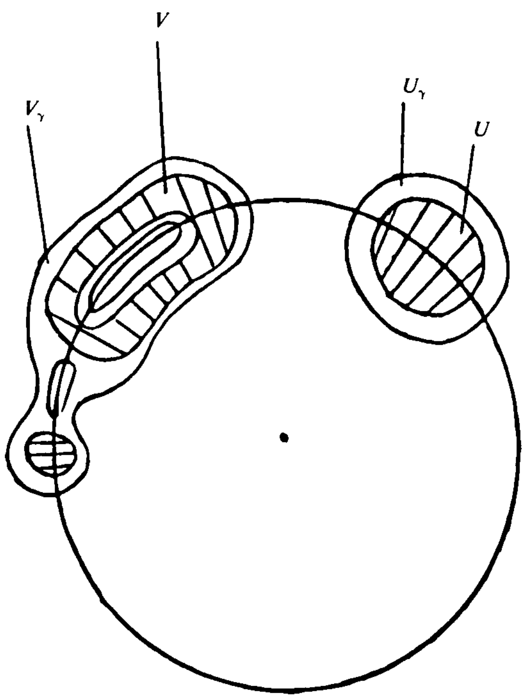

Fig. 2. Case where $\bar{G}|_U \ne G$.

Therefore there exists $A = A(\gamma, U_\gamma)$ such that

$$|H''(z)| \leq A,$$

$\forall z \in U$, $\forall H \in \mathcal{H}$.

The proposition then follows from Lemma 3.1(3).

4. Markov partitions

Let $g : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a finite Blaschke product. We write $g^* = g|_{\mathbf{S}^1}$.

DEFINITION 4.1. A partition $\mathcal{P} = \{I_1, \ldots, I_p\}$ of $\mathbf{S}^1$ into intervals is a *Markov partition* with respect to g^* if for every branch $\phi : I_j \to \mathbf{S}^1$ of $(g^*)^{-1}$ we have:

$$\phi(I_j) \cap I_i \neq \varnothing \Rightarrow \phi(I_j) \subset I_i,$$

$\forall 1 \leq i, j \leq p$.

DEFINITION 4.2. Let $\mathcal{P} = \{I_1, \ldots, I_p\}$ be a Markov partition with respect to g^*. We say that $\mathcal{P}$ is *compatible with* g if there are disks U_j, $1 \leq j \leq p$, symmetric with respect to $\mathbf{S}^1$, satisfying:

(1) $U_j \cap \mathbf{S}^1 = I_j$, $\forall 1 \leq j \leq p$, and

(2) If V is a connected component of $g^{-1}(U_j)$, then

$$V \cap I_i \neq \varnothing \Rightarrow V \cap \mathbf{S}^1 \subset I_i,$$

$\forall 1 \leq i, j \leq p$.

The disks U_j are said to be *associated* to I_j, $1 \leq j \leq p$.

DEFINITION 4.3. Let $\mathcal{P} = \{I_1, \ldots, I_p\}$ be a Markov partition with respect to g^* and compatible with g. Let B_j, $1 \leq j \leq r$ be the atoms of $\mathcal{P}^{(n)} := \mathcal{P} \vee \cdots \vee g^{-n}\mathcal{P}$ such that $g^n(B_j) = I_i$ with $i \neq 1$ and $i \neq p$. We say that $\mathcal{P}$ has *bounded distortion* if there exists a constant A independent of n such that

$$\sum_{j=1}^{r} \sup_{z \in I_i} |(T_{I_i B_j} g^n)'(z)| \leq A.$$

We recall that $T_{AB}F$ denotes the transition function associated to A, B and F (see section 2).

PROPOSITION 4.4. *There exist Markov partitions with respect to g^*, compatible with g and with bounded distortion.*

The rest of this section is devoted to the construction of these partitions.

Fix any $z_1 \in \mathbf{D}$ which is not a critical value of g. Let a_i, $1 \le i \le k+1$, be the zeros of $g - z_1$ ordered according to $0 \le |a_1| \le \cdots \le |a_{k+1}| < 1$. Let ζ_i, $1 \le i \le k$, be the fixed points of g^* ordered according to the trigonometric direction. Consider curves C_i, $1 \le i \le k$, C^1-near to the lines joining ζ_i to z_1, and such that they don't contain critical values of g. Let L_i, $1 \le i \le k$, be the lifting of C_i by g having ζ_i as base point and let $a_{s(i)}$ be the end point of this lifting.

Lemma 4.5. *Suppose that* $0 \le |a_1| \le |a_2| < R < 1$. *Then there exists* i_0, $1 \le i_0 \le k$, *such that* L_{i_0} *intersects* $\mathbf{D}_R$.

Proof. Suppose that there are indexes $i_1 < \cdots < i_{2m}$ such that $s(i_1) = s(i_2), \ldots, s(i_{2m-1}) = s(i_{2m})$. Let F_j, $1 \le j \le m$, be the curves $L_{i_{2j-1}} \vee L_{i_{2j}}^{-1}$ and E_j, $1 \le j \le m$, be the arcs of the circle joining $\zeta_{i_{2j}}$ to $\zeta_{i_{2j+1}}$. Consider the closed curve $C = F_1 \vee E_1 \vee \cdots \vee F_m \vee E_m$ and denote by S the region interior to this curve. Suppose also that S is minimal in the sense that there doesn't exist a proper subset of S that is the interior of a curve constructed as above. We shall calculate now the number of zeros of $g - z_1$ in the region S.

On a curve F_j there is exactly one zero of $g - z_1$, $p_j = a_{s(i_{2j-1})} = a_{s(i_{2j})}$. Modify the curve F_j in a neighborhood of p_j in the following way:

Let D_j be a small simple curve in the intersection of a neighborhood of p_j with the exterior of S, joining a point of $L_{i_{2j-1}}$ to a point of $L_{i_{2j}}$ (see Fig. 3). Let F_j' be the curve obtained from F_j replacing the part between these points by D_j. Consider the curve $C' = F_1' \vee E_1 \vee \cdots \vee F_m' \vee E_m$ and denote by S' its interior.

The number of zeros of $g - z_1$ in S is equal to the number of zeros of $g - z_1$ in S' minus m. The number of zeros of $g - z_1$ on S' is equal to the index of $g(C')$ around z_1, which is precisely

$$(*) \qquad 1 + \sum_{j=1}^{m} n_j + m$$

where n_j denotes the number of fixed points of g^* in E_j.

We shall explain this formula now. Each arc E_j is mapped by g onto a curve which starts at $\zeta_{i_{2j}}$, gives $1 + n_j$ complete turns around the circle and then ends at $\zeta_{i_{2j+1}}$. And a curve F_j' is mapped by g onto a curve G_j joining $\zeta_{i_{2j-1}}$ to $\zeta_{i_{2j}}$ without self-intersections. Moreover, we can see that $G_1 \vee E_1 \vee \cdots \vee G_m \vee E_m$ is a closed curve containing z_1 in its interior. This proves that the index of $g(C')$ around z_1 is given by $(*)$.

On the other hand, the liftings L_i starting at $\zeta_i \in E_j$, $1 \le j \le m$, must end at *distinct* zeros of $g - z_1$ belonging to S, by the minimality of S. Hence the number of

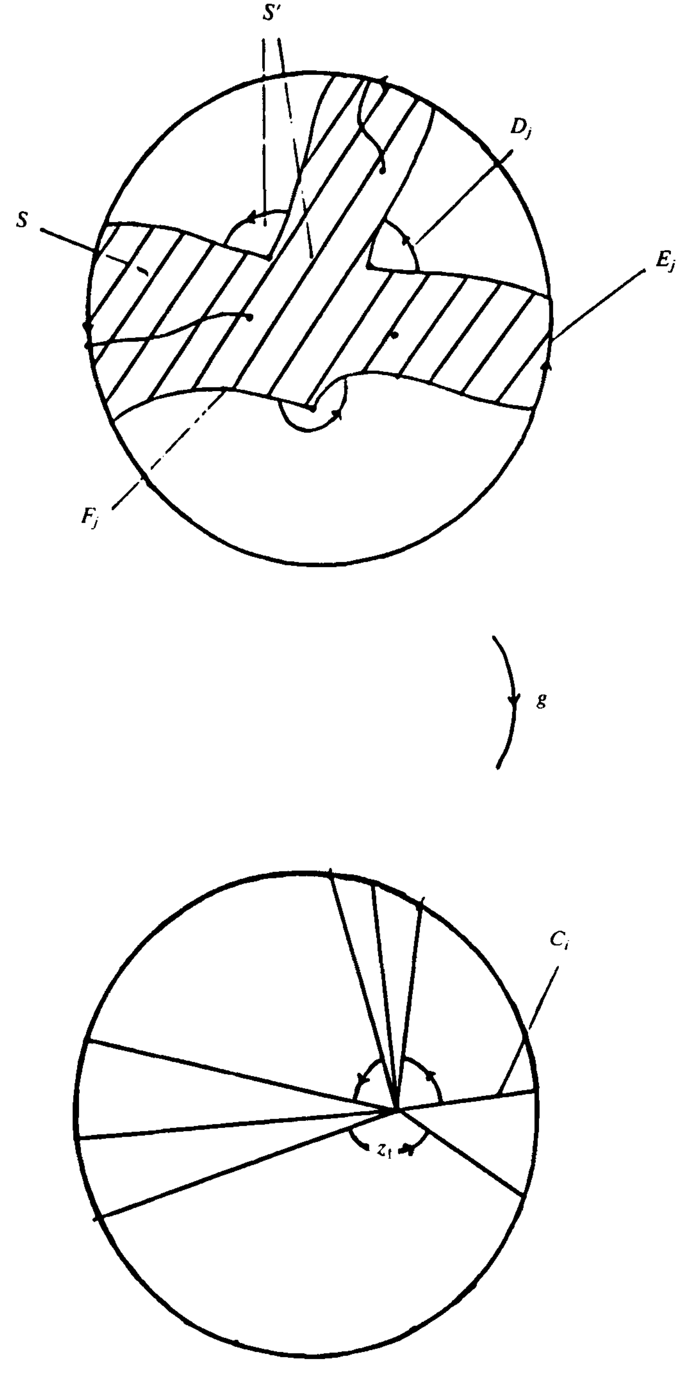

Fig. 3.

zeros of $g - z_1$ in S that are end points of some lifting L_i is $\sum_{j=1}^{m} n_j$. Comparing this with the number of zeros of $g - z_1$ in S' we conclude that there is exactly one zero of $g - z_1$ in S that is not an end point of some lifting L_i.

Suppose now that the whole disk $\mathbf{D}_R$ is contained in a region S like above. Then there exists a lifting L_{i_0} that ends at a_0 or a_1. In particular, this lifting intersects $\mathbf{D}_R$.

On the other hand, if $\mathbf{D}_R$ is not contained in any region S like above, it is clear that a lifting L_{i_0} intersects $\mathbf{D}_R$.

This proves Lemma 4.5.

Fix $0 \leq |a_1| \leq |a_2| < R < 1$. We shall denote by ξ_1 the point ζ_{i_0} and by C the curve C_{i_0} obtained in Lemma 4.5.

Let ξ_i, $1 \leq i \leq p$, be some pre-images of ξ_1 by g^* such that the liftings L_i of C by g having ξ_i as base point intersect the disk $\mathbf{D}_R$. Consider the partition $\mathcal{P} = \{I_1, \ldots, I_p\}$, where $I_i = [\xi_i, \xi_{i+1}]$, if $1 \leq i \leq p-1$, and $I_p = [\xi_p, \xi_1]$.

Proposition 4.6. *$\mathcal{P}$ is a Markov partition with respect to g^* and compatible with g.*

Proof. It can immediately be seen that $\mathcal{P}$ is a Markov partition with respect to g^*. Let us verify the compatibility with g.

Write $r = (2 + R)/3$ and define

$$U_i = \{z \in \mathbf{C} \,|\, r < |z| < r^{-1}, \arg \xi_i < \arg z < \arg \xi_{i+1}\},$$

$\forall 2 \leq i \leq p-1$. Let $\eta(t) = (1-t)e^{i\theta(t)}$, $t \in [0,1]$, be a parametrization of C. Define also

$$U_1 = \{z \in \mathbf{C} \,|\, r < |z| < r^{-1}, \theta(|z|) < \arg z < \arg \xi_2\}$$

and

$$U_p = \{z \in \mathbf{C} \,|\, r < |z| < r^{-1}, \arg \xi_p < \arg z < \theta(|z|)\}.$$

Let V be a connected component of $g^{-1}(U_i)$. Assume, to obtain a contradiction, that $V \cap I_{i_1} \neq \emptyset$ and $V \cap I_{i_2} \neq \emptyset$, where i_1 and i_2 are distinct indexes. Then there exists a curve in V joining a point of I_{i_1} to a point of I_{i_2}. Observe that, by the Schwarz lemma, V doesn't intersect $\mathbf{D}_R$ and hence this curve must intersect L_{i_3}, for some index i_3. But a point of intersection of these curves must be in

$V \subset g^{-1}(U_i)$ and in $L_{i_3} \subset g^{-1}(C)$. This is impossible, since by construction $U_i \cap C = \emptyset$ (see Fig. 4). Thus $\mathcal{P}$ is compatible with g.

Let $\alpha \in \mathbf{S}^1$ be such that $\arg \xi_1 < \arg \alpha < \arg \xi_2$ and $g(\alpha) = \xi_{i_0}$, for some $1 \leq i_0 \leq p$. Consider the partition $\mathcal{P}_\alpha = \{J_0, J_1, \ldots, J_p\}$, where $J_0 = [\xi_1, \alpha]$, $J_1 = [\alpha, \xi_2]$ and $J_i = I_i$, $\forall 2 \leq i \leq p$.

PROPOSITION 4.7. *Suppose that there exists a curve C_1 starting at ξ_{i_0}, intersecting $\mathbf{D}_R$ and such that its lifting by g having α as base point, denoted by L_1, intersects $\mathbf{D}_R$. Then $\mathcal{P}_\alpha$ is a Markov partition with respect to g^* and compatible with g.*

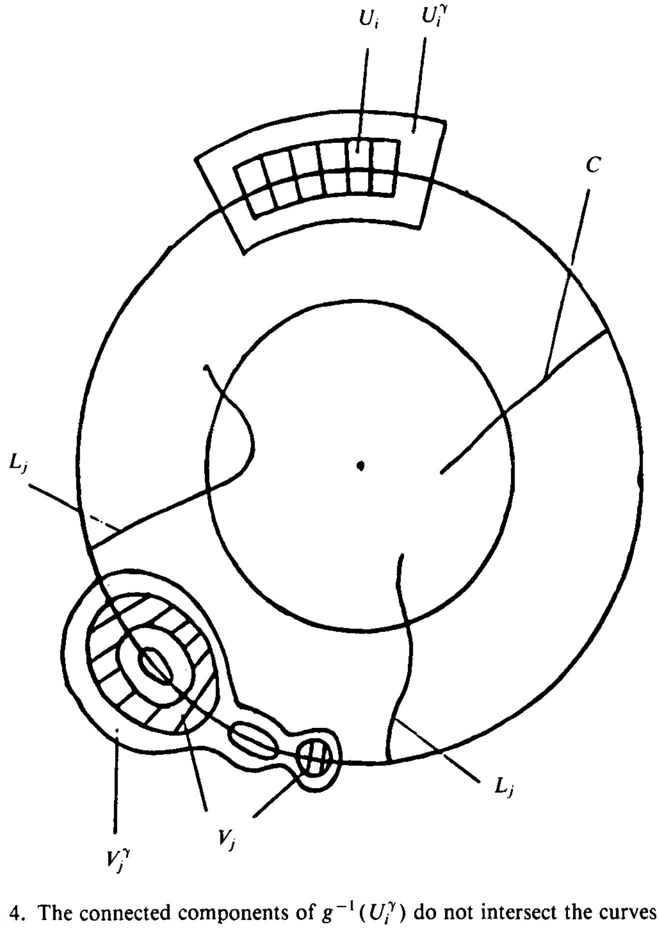

Fig. 4. The connected components of $g^{-1}(U_i^\gamma)$ do not intersect the curves L_j.

Proof. Define

$$W_0 = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \theta(|z|) < \arg z < \arg \alpha\},$$

$$W_1 = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \arg \alpha < \arg z < \arg \xi_2\},$$

$$W_{i_0-1} = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \arg \xi_{i_0-1} < \arg z < \theta_1(|z|)\}$$

and

$$W_{i_0} = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \theta_1(|z|) < \arg z < \arg \xi_{i_0+1}\},$$

where $\eta_1(t) = (1-t)\cdot e^{i\theta_1(t)}$, $t \in [0,1]$, is a parametrization of C_1. Define also $W_i = U_i$, for the other indexes i. We shall show that $\mathcal{P}_\alpha$ is a Markov partition with respect to g^* compatible with g with associated disks W_i, $0 \le i \le p$.

Since $\mathcal{P}$ is a Markov partition with respect to g^* compatible with g, we must only verify that if V is a connected component of $g^{-1}(W_i)$ such that $V \cap \mathbf{S}^1 \subset [\xi_1, \xi_2]$, then in reality $V \cap \mathbf{S}^1 \subset [\xi_1, \alpha]$ or $V \cap \mathbf{S}^1 \subset [\alpha, \xi_2]$.

Observe that we took W_i as a slight modification of U_i in such a way that it doesn't intersect the curve C_1. If $V \cap [\xi_1, \alpha] \ne \varnothing$ and $V \cap [\alpha, \xi_2] \ne \varnothing$, since V doesn't intersect $\mathbf{D}_R$, and using the hypothesis, we conclude that $V \cap L_1 \ne \varnothing$. Hence there would exist a point in V whose image would be in C_1, which is a contradiction.

This proves Proposition 4.7.

Lemma 4.8. *Let $\mathcal{R}$ be a partition of $\mathbf{S}^1$ into intervals. Fix $I_i \in \mathcal{R}$, $B_j \in \mathcal{R}^{(n)}$ and let U_i be an open set such that $U_i \cap \mathbf{S}^1 = I_i$. Consider a holomorphic function $G_{n,i,j}: U_i \to \mathbf{C}$ whose real part is given by*

$$\operatorname{Re} G_{n,i,j}(z) = \sum_{\substack{g^n(w)=z \\ w \in V_j}} \log|w|,$$

where V_j is the union of the connected components of $g^{-n}(U_i)$ that contain B_j. Then

$$T_{I_i B_j} g^n(z) = |G'_{n,i,j}(z)|,$$

$\forall z \in I_i$. Consider a holomorphic function $\bar{G}_{n,i,j}: U_i^\gamma \to \mathbf{C}$ whose real part is given by

$$\operatorname{Re} \bar{G}_{n,i,j}(z) = \sum_{\substack{g^n(w)=z \\ w \in V_j^\gamma}} \log|w|,$$

where U_i^γ is a γ-extension of U_i and V_j^γ is the union of the connected components of $g^{-n}(U_i^\gamma)$ that contain B_j. If $\bar{G}_{n,i,j}|_{U_i} = G_{n,i,j}$, then

$$|(T_{I_i B_j} g^n)'(z)| \leq A\lambda(V_j^\gamma \cap \mathbf{S}^1),$$

$\forall z \in I_i$, where A is a constant that depends only on the regions U_i and U_i^γ.

PROOF. Indeed,

$$|G'_{n,i,j}(z)| = \sum_{\substack{g^n(w)=z \\ w \in B_j}} \frac{1}{|(g^n)'(w)|} = T_{I_i B_j} g^n(z),$$

$\forall z \in I_i$. The second claim is a consequence of Proposition 3.4.

LEMMA 4.9. *Let $\mathcal{P}$ be a partition as in Proposition 4.6. If $2 \leq i \leq p-1$, there exists a γ-extension U_i^γ of U_i such that $\bar{G}_{n,i,j}|_{U_i} = G_{n,i,j}$.*

PROOF. We write $r_1 = (1+2R)/3$ and $d_0 = \min\{|I_i|;\ 1 \leq i \leq p\}$. Define

$$U_i^\gamma = \{z \in \mathbf{C} \mid r_1 < |z| < r_1^{-1}, \arg \xi_i - d_0/2 < \arg z < \arg \xi_{i+1} + d_0/2\},$$

$\forall 2 \leq i \leq p-1$. Then U_i^γ is a γ-extension of U_i, for a certain $\gamma = \gamma(R, d_0)$. Besides, U_i^γ doesn't intersect C. So we are sure that each connected component of $g^{-n}(U_i^\gamma)$ intersected with $\mathbf{S}^1$ is a subset of an atom of $\mathcal{P}$. It follows then that the holomorphic functions $\bar{G}_{n,i,j} : U_i^\gamma \to \mathbf{C}$ whose real parts are given by

$$\operatorname{Re} \bar{G}_{n,i,j}(z) = \sum_{\substack{g^n(w)=z \\ w \in V_j^\gamma}} \log|w|$$

are really extensions of $G_{n,i,j}$, proving the lemma.

PROPOSITION 4.10. *The partitions $\mathcal{P}$ have bounded distortion.*

PROOF. It follows from Lemmas 4.8 and 4.9 that if $2 \leq i \leq p-1$, then

$$|(T_{I_i B_j} g^n)'(z)| \leq A\lambda(V_j^\lambda \cap \mathbf{S}^1),$$

$\forall z \in I_i$. Equivalently

$$\sup_{z \in I_i} |(T_{I_i B_j} g^n)'(z)| \leq A\lambda(V_j^\gamma \cap \mathbf{S}^1).$$

Since each point of $\mathbf{S}^1$ can be, at most, an element of two U_i^γ, $1 \le i \le p$, the same is true for the V_j^γ, $1 \le j \le s$. Hence

$$\sum_{j=1}^{r} \lambda(V_j^\gamma) \le 2$$

and therefore

$$\sum_{j=1}^{r} \sup_{z \in I_i} |(T_{I_i B_j} g^n)'(z)| \le 2A,$$

proving that the partitions $\mathcal{P}$ constructed above have bounded distortion.

Proposition 4.11. *Consider a partition $\mathcal{P}_\alpha$ as in Proposition 4.7. Let Z be a neighborhood of L_1 without critical points. Then $\mathcal{P}_\alpha$ has bounded distortion with a constant that depends on $g(Z)$.*

Proof. We prove first that if $2 \le i \le p-1$, there exists a γ-extension W_i^γ of W_i such that $\bar{G}_{n,i,j}|_{W_i} = G_{n,i,j}$. Let $d_0 = \min\{|J_i|; 0 \le i \le p\}$. Define

$$W_1^\gamma = \{z \in \mathbf{C} \,|\, r_1 < |z| < r_1^{-1}, \arg\alpha - d_0/2 < \arg z < \arg\xi_2 + d_0/2\},$$

$$W_{i_0-1}^\gamma = \{z \in \mathbf{C} \,|\, r_1 < |z| < r_1^{-1}, \arg\xi_{i_0-1} - d_0/2 < \arg z < \arg\xi_{i_0} + d/2\}$$

and

$$W_{i_0}^\gamma = \{z \in \mathbf{C} \,|\, r_1 < |z| < r_1^{-1}, \arg\xi_{i_0} - d/2 < \arg z < \arg\xi_{i_0+1} + d_0/2\},$$

where d is such that $g(Z) \supset X := \{z \in \mathbf{C} \,|\, r_1 < |z| < r_1^{-1}, \arg\xi_{i_0} - d < \arg z < \arg\xi_{i_0} + d\}$. For the other indexes take $W_i^\gamma = U_i^\gamma$, as in Lemma 4.9. Then W_i^γ is a γ-extension of W_i, $\forall 1 \le i \le p-1$, where $\gamma = \gamma(R, d_0, d)$. And if $i \ne i_0$ and $i \ne i_0 - 1$ we prove as in Lemma 4.9 that $\bar{G}_{n,i,j}|_{W_i} = G_{n,i,j}$.

The difficulty that appears in this case is with the indexes $i_0 - 1$ and i_0 (recall that $g(\alpha) = \xi_{i_0}$). The extensions $W_{i_0-1}^\gamma$ and $W_{i_0}^\gamma$ intersect the curve C_1 and hence a connected component of V_j^γ of $g^{-1}(W_{i_0-1}^\gamma)$ or $g^{-1}(W_{i_0}^\gamma)$ that intersects $[\alpha, \xi_2]$ can also intersect $[\xi_1, \alpha]$.

But since the connected components of $g^{-1}(X)$ are simply connected, we have that, in the first case, $g(V_j^\gamma \cap [\alpha, \xi_2]) \subset I_{i_0}$, while in the second case, $g(V_j^\gamma \cap [\xi_1, \alpha]) \subset I_{i_0-1}$.

This proves that the functions $\bar{G}_{n,i,j}: W_i^\gamma \to \mathbf{C}$ are really extensions of $G_{n,i,j}$, if $i = i_0$ or $i = i_0 - 1$.

Applying then the same reasoning used for the partition $\mathcal{P}$ in Proposition 4.10 we prove that $\mathcal{P}_\alpha$ has bounded distortion, with a constant that depends on $g(Z)$.

5. Limit behavior of the Markov partitions associated to approximations of f

Let f be an inner function with $f(0) = 0$. It is well known that there exists a sequence (f_k) of finite Blaschke products converging uniformly to f on compact subsets of $\mathbf{D}$. Let C_k be the basic curve used in the construction of Markov partitions with respect to f_k^* compatible with f_k, which was done in section 4. And let $L_{i,k}$, $1 \le i \le \text{degree}(f_k)$, be the liftings of C_k by f_k. Define

$$\mathfrak{L}_{R,k} = \{L_{i,k} \mid L_{i,k} \cap \mathbf{D}_R \neq \emptyset\}.$$

LEMMA 5.1. *There exists $N = N(R) \in \mathbf{N}$ such that $s_k := \#\mathfrak{L}_{R,k} \le N$.*

PROOF. Write $\eta(t) = \lim_{k\to\infty} \eta_k(t)$, where $\eta_k(t)$ is a parametrization of C_k. Then η is a parametrization of a curve C, C^1-near to the line joining ω_1 to z_1. We are considering that z_1 is the end point of the curve C_k, $\forall k \in \mathbf{N}$, and that the starting points $\omega_{1,k}$ are converging to ω_1.

Take $0 < R < 1$ such that $f(\mathbf{S}_R)$ intersects C transversally, where $\mathbf{S}_R = \{z \in \mathbf{D} \mid |z| = R\}$. Say $\eta(t_i) \in f(\mathbf{S}_R)$, $1 \le i \le l$. Write $A_i = (t_i, t_{i+1})$, $1 \le i \le l-1$, $A_0 = (0, t_1)$, and $A_l = (t_l, 1)$. Then, for k sufficiently large, C_k intersects $f_k(\mathbf{S}_R)$ transversally at the points $\eta(t_{i,k})$, $1 \le i \le l$, forming the intervals $A_{i,k} = (t_{i,k}, t_{i+1,k})$, $1 \le i \le l-1$, $A_{0,k} = (0, t_{1,k})$ and $A_{l,k} = (t_{l,k}, 1)$.

Consider the function $N_{R,k} : \mathbf{D} \backslash f_k(\mathbf{S}_R) \to \mathbf{N}$ given by

$$N_{R,k}(\zeta) = \frac{1}{2\pi i} \int_{\mathbf{S}_R} \frac{f_k'(z)}{f_k(z) - \zeta}\, dz,$$

which indicates the number of zeros of $f_k - \zeta$ on $\mathbf{D}_R$. Consider also the function $N_k : \bigcup_{0 \le i \le l} A_{i,k} \to \mathbf{N}$ given by

$$N_k(t) = N_{R,k}(\eta_k(t)).$$

Then N_k is constant in each interval $A_{i,k}$ and assumes the value m_i, independent of k. Hence

$$\#\{L_{j,k} \mid \alpha_{j,k}(A_{i,k}) \cap \mathbf{D}_R \neq \emptyset\} = m_i,$$

where $\alpha_{j,k}$ is a parametrization of $L_{j,k}$, and therefore

$$s_k = \#\mathfrak{L}_{R,k} \le \sum_{i=1}^{l} m_i = N.$$

Let $\omega_{j,k}(R)$, $1 \le j \le s_k(R)$ be the starting point of $L_{j,k} \in \mathfrak{L}_{R,k}$. It follows from Lemma 5.1 that taking subsequences we can assume that $s_k(R) = s(R)$, if

$k \geq k_0(R)$, and that $\omega_{j,k}(R)$ converges to $\omega_j(R)$, if $1 \leq j \leq s(R)$. Write $\Omega(R) = \{\omega_j(R);\ 1 \leq j \leq s(R)\}$.

If $R_1 < R_2$, then $s(R_1) \leq s(R_2)$ and $\Omega(R_1) \subset \Omega(R_2)$. Moreover, $s(R)$ tends to infinity when R tends to 1. The subsequence of (f_k) that we use for defining $\Omega(R)$ depends on R. But, using the diagonal process, we can use the same subsequence for defining $\Omega(R_j)$, where (R_j) is a sequence tending to 1. Write $\Omega = \bigcup \Omega(R_j)$.

PROPOSITION 5.2. *Let $I \subset \mathbf{S}^1$ be an interval such that $I \cap \Omega = \emptyset$. Then f^* is analytic and injective in I.*

For the proof of this proposition, we shall need the following theorem, due to Frostman, which can be found in [Co, p. 50].

THEOREM 5.3. *Let f be an inner function. There exists a set $E(f) \subset \mathbf{D}$ of zero capacity such that if $\xi \notin E(f)$ then $T_\xi \circ f$ is a Blaschke product, where*

$$T_\xi(z) = \frac{z - \xi}{1 - \bar{\xi} z}.$$

PROOF OF PROPOSITION 5.2. Observe first that, if $z_1 \notin E(f)$, then f^* is analytic at $x \in \mathbf{S}^1$ if and only if the sequence of pre-images of z_1, denoted by (a_j), doesn't accumulate at x.

Suppose that f^* is not analytic at $x \in I$. Then there exists a subsequence of (a_j) converging to x. We shall denote this subsequence by the same indexes of the original sequence.

Fix $0 < R < 1$.

CLAIM. We can assume that $L_{j,k} \in \mathcal{L}_{R,k}$ for only a finite number of values of k.

Indeed, suppose that j_0 is such that $L_{j_0,k} \in \mathcal{L}_{R,k}$ for infinite values of k. Then taking subsequences of (f_k) we can assume that $L_{j_0,k} \in \mathcal{L}_{R,k}$ for all values of k. Therefore, it follows from Lemma 5.1 that

$$\#\{j \neq j_0 \,|\, L_{j,k} \in \mathcal{L}_{R,k}\} \leq N - 1.$$

We exclude then a_{j_0} from the sequence (a_j). If there exists another j_1 such that $L_{j_1,k} \in \mathcal{L}_{R,k}$ for infinite values of k, we repeat the procedure and exclude a_{j_1} from the sequence (a_j). It is clear that after at most N steps of this procedure we will arrive at a situation where for each $j \in \mathbf{N}$, $L_{j,k} \in \mathcal{L}_{R,k}$ for only a finite number of values of k.

This proves the claim.

Then, given j, there exists $k_0 = k_0(j)$ such that $L_{j,k} \notin \mathcal{L}_{R,k}$, if $k > k_0$. And the liftings $L_{j,k}$ end at points $a_{j,k}$ satisfying $\lim_{k\to\infty} a_{j,k} = a_j$. Since $\omega \notin I$, $\forall\omega \in \Omega$, we can assume that $L_{j,k}$ intersects C_{J_1} (or C_{J_2}), if $k > k_0$, where we have used the following notation:

I is the interval (a,b), J_1 is an interval contained in (x,b); J_2 is an interval contained in (a,x) and

$$C_J = \{re^{i\theta} \mid 0 \le r \le 1, e^{i\theta} \in J\}.$$

We conclude that if $e^{i\theta} \in J_1$ (or J_2), there exists $z_k = |z_k|e^{i\theta}$, with $|z_k| > R$ such that $f_k(z_k) \in C_k$. It follows then from Proposition 5.5, proved below, that $f^*(e^{i\theta}) = \omega_1$ for a.e. $e^{i\theta} \in J_1$. This contradicts the fact that $\lambda((f^*)^{-1}(x)) = 0$, $\forall x \in \mathbf{S}^1$. Hence f^* is analytic in I.

If f^* were not injective in I, there would exist $\zeta \in I$ such that $f^*(\zeta) = \omega_1$. By analytic continuation, there would exist $\zeta_k \in I$, ζ_k converging to ζ and such that $f_k(\zeta_k) = \omega_{1,k}$. Moreover, the lifting of C_k by f_k having ζ_k as base point would intersect $\mathbf{D}_R$, for some $0 < R < 1$ independent of k. Hence $\zeta \in \Omega$, contradicting the hypothesis.

This proves the proposition.

Lemma 5.4. *Let f be an inner function with $f' \in N$. There exists a sequence (f_k) of finite Blaschke products converging to f uniformly on compact subsets of* $\mathbf{D}$ *such that*

$$L = \sup_k \int_{\mathbf{S}^1} \log|f_k'|\, d\lambda < \infty.$$

Proof. By Theorem 5.3, we can choose $\xi \in \mathbf{D}$ such that $T_\xi \circ f$ is a Blaschke product. Choose a sequence (f_k) of finite Blaschke products converging to f uniformly on compact subsets of $\mathbf{D}$ and such that $(T_\xi \circ f_k)$ is a sequence of partial products of $T_\xi \circ f$. Then

$$|(T_\xi \circ f_k)'(x)| \le |((T_\xi \circ f)')^*(x)|,$$

$\forall x \in \mathbf{S}^1$, and therefore

$$\begin{aligned} |(f_k)'(x)| &\le \frac{1+|\xi|}{1-|\xi|}\, |((T_\xi \circ f)')^*(x)| \\ &\le \left[\frac{1+|\xi|}{1-|\xi|}\right]^2 |(f')^*(x)|. \end{aligned}$$

Hence

$$L \leq 2\log\frac{1+|\xi|}{1-|\xi|} + \int_{\mathbf{S}^1} \log|(f')^*|\,d\lambda < \infty,$$

proving the lemma.

PROPOSITION 5.5. *Let f be an inner function with $f' \in N$. Let (f_k) be a sequence of finite Blaschke products converging uniformly on compact subsets of* $\mathbf{D}$ *to f and satisfying the property of Lemma* 5.4. *Then for a.e. $e^{i\theta} \in \mathbf{S}^1$ the following is true*:

Given any sequence (z_k) of points in $\mathbf{D}$ *with* $\arg(z_k) = \theta$ *and converging to $e^{i\theta}$, there exists a subsequence (k_j) such that $f_{k_j}(z_{k_j})$ converges to $f^*(e^{i\theta})$.*

The rest of section 5 is devoted to the proof of this proposition.

Given $\alpha > 1$, consider the Stoltz angle at $e^{i\theta}$

$$\Gamma_\alpha(e^{i\theta}) = \left\{ z \in \mathbf{D} \,\middle|\, \frac{|e^{i\theta} - z|}{1 - |z|} < \alpha \right\}.$$

If u is a function of $\mathbf{D}$ define

$$u^\#(e^{i\theta}) = \sup_{z \in \Gamma_\alpha(e^{i\theta})} |u(z)|.$$

PROPOSITION 5.6. *Let f be an inner function with $f' \in N$. Let (f_k) be a sequence of finite Blaschke products converging uniformly to f on compact subsets of* $\mathbf{D}$ *and satisfying the property of Lemma* 5.4. *Then for a.e. $e^{i\theta} \in \mathbf{S}^1$ there exists a subsequence (k_j) such that*

$$\sup_j (|f'_{k_j}|)^\# < \infty.$$

Let us first see how Proposition 5.5 follows from Proposition 5.6.

Take the subsequence (k_j) given by Proposition 5.6. We can suppose that $(f_{k_j}(e^{i\theta}))$ is converging to $f^*(e^{i\theta})$ (see Lemma 6.1). Given $\epsilon > 0$, take $j_0 > 0$ such that if $j > j_0$ then

$$|f^*(e^{i\theta}) - f_{k_j}(e^{i\theta})| < \epsilon/2$$

and also

$$1 - |z_{k_j}| < \frac{\epsilon}{2} \sup_j (|f'_{k_j}|)^\#.$$

It follows that

$$|f^*(e^{i\theta}) - f_{k_j}(z_{k_j})| \le |f^*(e^{i\theta}) - f_{k_j}(e^{i\theta})| + |f_{k_j}(e^{i\theta}) - f_{k_j}(z_{k_j})|$$

$$\le \frac{\epsilon}{2} + (1 - |z_{k_j}|)\sup_j (|f'_{k_j}|)^\# < \epsilon,$$

proving Proposition 5.5.

Let us now prove Proposition 5.6.

PROPOSITION 5.7. *Let g be a holomorphic function of* $\bar{\mathbf{D}}$ *such that* $|g(x)| \ge 1$, $\forall x \in \mathbf{S}^1$. *Then*

$$\lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid |g|^\#(e^{i\theta}) > \beta\}) \le \frac{A_\alpha}{\log\beta}\frac{1}{2\pi}\int_0^{2\pi}\log|g(e^{it})|\,dt$$

where $\alpha > 1$, $\beta > 1$ *and* A_α *is a constant that depends only on* α.

Proposition 5.6 follows from Proposition 5.7 in the following way:
Write

$$E_{\beta,k_0} = \{e^{i\theta} \in \mathbf{S}^1 \mid |f'_k|^\#(e^{i\theta}) > \beta,\ \forall \beta > k_0\}$$

and

$$L = \sup_k \frac{1}{2\pi}\int_0^{2\pi}\log|f'_k(e^{it})|\,dt.$$

By hypothesis, L is a finite number. Therefore, it follows from Proposition 5.7 that

$$\lambda(E_{\beta,k_0}) \le L\frac{A_\alpha}{\log\beta}$$

and hence, since the sets $E_{\beta,k}$ are increasing with k,

$$\lambda\left(\bigcup_{k\in\mathbf{N}} E_{\beta,k}\right) \le L\frac{A_\alpha}{\log\beta}.$$

But if $e^{i\theta} \notin (\bigcup_{k\in\mathbf{N}} E_{\beta,k})$, then there exists a subsequence (k_j) such that

$$\sup_j |f'_{k_j}|^\#(e^{i\theta}) < \infty.$$

Making β tend to infinity, Proposition 5.6 is proved.

The problem is now reduced to proving Proposition 5.7. For this, we need to introduce the following notion:

Let h be an integrable function on $\mathbf{S}^1$. The *Hardy-Littlewood maximal function* of h is

$$Mh(e^{i\theta}) := \sup \frac{1}{\lambda(I)} \int_I |h(e^{it})| \, dt$$

where the sup is taken over the intervals I that contain $e^{i\theta}$ in its interior.

We shall use two theorems concerning maximal functions, namely:

THEOREM 5.8. *Let u be a harmonic function on $\bar{D}$. Then there exists a constant A_α, depending only on α, such that*

$$u^{\#}(e^{i\theta}) \leq A_\alpha Mu(e^{i\theta}).$$

THEOREM 5.9. *Suppose that $h \in \mathcal{L}^1(\mathbf{S}^1)$. Then, for any $\beta > 0$,*

$$\lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid Mh(e^{i\theta}) > \beta\}) \leq \frac{2}{\beta} \|h\|_1.$$

The proof of these theorems can be found in [G, pp. 22–25]. Let us prove Proposition 5.7.

Let $g = B \cdot g_1$, where B is a finite Blaschke product and g_1 has no zeros in $\mathbf{D}$. Since $|g| < |g_1|$ in $\mathbf{D}$,

$$\begin{aligned}\lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid |g|^{\#}(e^{i\theta}) > \beta\}) &\leq \lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid |g_1|^{\#}(e^{i\theta}) > \beta\}) \\ &= \lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid \log|g_1|^{\#}(e^{i\theta}) > \log\beta\}).\end{aligned}$$

Applying Theorem 5.8 to the harmonic function $\log|g_1|$ we obtain

$$\lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid |g|^{\#}(e^{i\theta}) > \beta\}) \leq \lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid A_\alpha M \log|g_1|(e^{i\theta}) > \log\beta\}).$$

But since $|g| = |g_1|$ in $\mathbf{S}^1$, $M\log|g_1|(e^{i\theta}) = M\log|g|(e^{i\theta})$. Hence it follows from Theorem 5.9 that

$$\lambda(\{e^{i\theta} \in \mathbf{S}^1 \mid |g|^{\#}(e^{i\theta}) > \beta\}) \leq \frac{2A_\alpha}{\log\beta} \frac{1}{2\pi} \int_0^{2\pi} \log|g(e^{it})| \, dt,$$

proving Proposition 5.7, since the constant 2 is not relevant.

6. Proof of Theorem 2.2

Let f be an inner function with $f(0) = 0$. Consider a sequence (f_k) of finite Blaschke products with $f_k(0) = 0$ converging uniformly to f on compact subsets of $\mathbf{D}$.

LEMMA 6.1. *The sequence (f_k^*) converges to f^* in the norm of the Hilbert space $\mathcal{L}^2(\mathbf{S}^1,\mathbf{C})$.*

PROOF. Let $(f_{k_l}^*)$ be a subsequence of (f_k^*) converging to a function F weakly in $\mathcal{L}^2(\mathbf{S}^1,\mathbf{C})$. We have, using the Poisson formula, that for any $z \in \mathbf{D}$,

$$\begin{aligned} f(z) &= \lim_{l\to\infty} f_{k_l}(z) \\ &= \lim_{l\to\infty} \frac{1}{2\pi}\int_0^{2\pi} P_z(t) f_{k_l}^*(e^{it})\,dt, \end{aligned}$$

and hence

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t) F(e^{it})\,dt.$$

Therefore, $F(e^{it}) = f^*(e^{it})$, for a.e. $t \in [0,2\pi]$. It results then, from the fact that the unit ball in $\mathcal{L}^2(\mathbf{S}^1,\mathbf{C})$ is weakly compact, that the sequence (f_k^*) converges to f^* weakly in $\mathcal{L}^2(\mathbf{S}^1,\mathbf{C})$. Denoting by $\langle\ ,\ \rangle$ the hermitian product of the Hilbert space $\mathcal{L}^2(\mathbf{S}^1,\mathbf{C})$, we have

$$\begin{aligned} \langle f^* - f_k^*, \overline{f^* - f_k^*}\rangle &= \langle f^*,\overline{f^*}\rangle + \langle f_k^*,\overline{f_k^*}\rangle - \langle f^*,\overline{f_k^*}\rangle - \langle f_k^*,\overline{f^*}\rangle \\ &= 2(1 - \operatorname{Re}\langle \overline{f^*}, f_k^*\rangle). \end{aligned}$$

But since the sequence (f_k^*) converges to f^* weakly, $(\langle \overline{f^*}, f_k^*\rangle)$ converges to 1 and therefore $(\|f^* - f_k^*\|_2)$ converges to 0.

Another proof of this lemma can be found in [W].

Since (f_k^*) converges to f^* in $\mathcal{L}^2(\mathbf{S}^1,\mathbf{C})$, there exists a subsequence converging a.e. In this section we shall always assume that (f_k^*) converges to f^* a.e.

Let $F_k, F\colon (X,\mathcal{A},\mu) \to (X,\mathcal{A},\mu)$ be endomorphisms. We say that (F_k) converges to F in measure if, for any $S \in \mathcal{A}$,

$$\lim_{k\to\infty} \mu(F_k^{-1}(S)\Delta F^{-1}(S)) = 0.$$

LEMMA 6.2. *Suppose that (F_k) converges to F a.e. Then (F_k) converges to F in measure.*

PROOF. Take $S \in \mathcal{A}$. Then

$$\mu(F_k^{-1}(S) \Delta F^{-1}(S)) = \int_X |\chi_{F_k^{-1}(S)} - \chi_{F^{-1}(S)}| \, d\mu$$

$$= \int_X |\chi_S \circ F_k - \chi_S \circ F| \, d\mu.$$

But $(|\chi_S \circ F_k - \chi_S \circ F|)$ converges to 0 a.e. The lemma follows by the dominated convergence theorem.

LEMMA 6.3. *Suppose that (F_k) converges to F in measure. Let $A_k, A, B_k, B \in \mathcal{A}$ be such that $\mu(A_k \Delta A) \to 0$ and $\mu(B_k \Delta B) \to 0$. Then $(T_{A_k B_k} F_k)$ converges to $T_{AB}F$ weakly, i.e., for any $S \subset A$,*

$$\lim_{k \to \infty} \int_S T_{A_k B_k} F_k \, d\mu = \int_S T_{AB} F \, d\mu.$$

PROOF. Take $S \subset A$. Since

$$|\mu(F^{-1}(S) \cap B) - \mu(F_k^{-1}(S) \cap B_k)| \leq \mu(F^{-1}(S) \Delta F_k^{-1}(S)) + \mu(B \Delta B_k),$$

we have

$$\lim_{k \to \infty} \mu(F_k^{-1}(S) \cap B_k) = \mu(F^{-1}(S) \cap B),$$

and therefore

$$\lim_{k \to \infty} \int_S T_{A_k B_k} F_k \, d\mu = \mu(F^{-1}(S) \cap B).$$

We conclude that

$$\lim_{k \to \infty} \int_S T_{A_k B_k} F_k \, d\mu = \int_S T_{AB} F \, d\mu.$$

COROLLARY 6.4. *Let I_k, I be intervals of $\mathbf{S}^1$ and B_k, B be Borel subsets of $\mathbf{S}^1$ such that $\lambda(I_k \Delta I) \to 0$ and $\lambda(B_k \Delta B) \to 0$. Then $(T_{I_k B_k} f_k^*)$ converges weakly to $T_{IB} f^*$.*

PROOF. Immediate from Lemmas 6.2 and 6.3.

DEFINITION 6.5. Let $\mathcal{P} = \{I_1, \ldots, I_p\}$ be a partition of $\mathbf{S}^1$ into intervals. Suppose that $\mathcal{P} = \lim_{k\to\infty} \mathcal{P}_k$, where $\mathcal{P}_k = \{I_{1,k}, \ldots, I_{p,k}\}$ are Markov partitions with respect to f_k^* and compatible with f_k. Suppose also that the open sets $U_{i,k}$ associated with the intervals $I_{i,k}$ (see Definition 4.2) are converging to open sets U_i containing I_i. We say then that $\mathcal{P}$ is a *Markov partition with respect to* f^* *compatible with* f.

REMARK. In the definition above, the open sets $U_{i,k}$ are converging to the open set U_i, $1 \le i \le p$, if for each compact set $B \subset U_i$, there exists $k_0 > 0$ such that $B \subset U_{i,k}$, if $k > k_0$.

LEMMA 6.6. *Let* $\mathcal{P} = \{I_1, \ldots, I_p\}$ *be a Markov partition with respect to* f^* *and compatible with* f. *Denote by* B_j, $1 \le j \le s$, *the atoms of* $\mathcal{P}^{(n)}$. *Then* $T_{I_i B_j}(f^*)^n$ *is real analytic*, $\forall n \in \mathbf{N}$, $\forall 1 \le i \le p$, $\forall 1 \le j \le s$.

PROOF. If $1 \le i \le p$, consider the open set $U_{i,k}$ associated with the interval $I_{i,k}$ of partition $\mathcal{P}_k$. Let $V_{j,k}$ be the union of the connected components of $f_k^{-n}(U_{i,k})$ that intersect $B_{j,k}$.

Since the Markov partition $\mathcal{P}_k$ is compatible with f_k, $V_{j,k} \cap \mathbf{S}^1 = B_{j,k}$. Hence, if we denote by $T_k : U_{i,k} \to \mathbf{C}$ a holomorphic function whose real part is given by

$$\operatorname{Re} T_k = \sum_{\substack{f_k(w)=z \\ w \in V_{j,k}}} \log|w|,$$

then $|T_k'|\,|_{I_{i,k}} = T_{I_{i,k}B_{j,k}} f_k^n$ (see Lemma 4.8).

But it follows from Proposition 3.2 that (T_k) is a normal family of holomorphic functions. In reality, these functions are defined in slightly different domains, but this causes no difficulty since their domains are converging to the open set U_i. (We can apply the reasoning to any open set W_i such that $\overline{W_i} \subset U_i$.)

Hence there exists a holomorphic function T defined on U_i that is the limit, uniform on compact subsets of U_i, of some subsequence of (T_k). And therefore, $(|T_k'|\,|_{I_{i,k}})$ is converging uniformly to $|T'|\,|_{I_i}$.

But we know from Corollary 6.4 that $(T_{I_{i,k}B_{j,k}} f_k^n)$ converges weakly to $T_{I_i B_j} f^n$, and hence $|T'|\,|_{I_i} = T_{I_i B_j} f^n$, proving the lemma.

DEFINITION 6.7. Let $\mathcal{P} = \{I_1, \ldots, I_p\}$ be a Markov partition with respect to f^* and compatible with f. We say that $\mathcal{P}$ has *bounded distortion* if the partitions $\mathcal{P}_k$ have bounded distortion with a constant A independent of k (see Definition 4.3).

LEMMA 6.8. *Let* $\mathcal{P}$ *be a partition with bounded distortion. Then* $\mathcal{P}$ *satisfies property* (P2) *of Theorem* 2.2.

PROOF. If we denote by $B_{j,k}$, $1 \le j \le r$, the atoms of $\mathcal{P}_k^{(n)}$ which are taken by $(f_k^*)^{(n)}$ onto $I_{i,k}$, where $i \ne 1$ and $i \ne p$, then

$$\sum_{j=1}^{r} \sup_{z \in I_{i,k}} |[T_{I_{i,k}B_{j,k}}(f_k^*)^n]'(z)| \le A.$$

Taking the limit in k, we have

$$\sum_{j=1}^{r} \sup_{z \in I_i} |[T_{I_i B_j}(f^*)^n]'(z)| \le A,$$

and hence

$$\sum_{j=1}^{r} \operatorname{osc} T_{I_i B_j}(f^*)^n \le A,$$

which proves the lemma.

Let us prove now Theorem 2.2. Consider the set Ω defined in section 5 and the distinguished element $\omega_1 \in \Omega$.

Case A. f^* is not analytic in any interval of the form (ω_1, b) or of the form (b, ω_1).

In this case, it follows from Proposition 5.2 that there exists $0 < R < 1$ such that $\Omega(R)$ contains points $\omega_2 \in (\omega_1, \omega_1 + \epsilon)$ and $\omega_3 \in (\omega_1 - \epsilon, \omega_1)$. Take a subset $\Xi(R)$ of $\Omega(R)$ containing ω_1, ω_2 and ω_3 and such that $\#\Xi(R) \cdot \epsilon \le 1$. Consider the partition $\mathcal{P}_\epsilon$ of $\mathbf{S}^1$ into intervals whose extremities are the points of $\Xi(R)$.

CLAIM A1. $\mathcal{P}_\epsilon$ satisfies property (P1) of Theorem 2.2.

$\mathcal{P}_\epsilon$ is the limit of the partitions $\mathcal{P}_k$, that are Markov partitions with respect to f_k^* compatible with f_k, by Proposition 4.6. Moreover, the open sets $U_{i,k}$ associated with the intervals of the partitions $\mathcal{P}_k$ (see Proposition 4.6) are converging to the open sets

$$U_i = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \arg \xi_i < \arg z < \arg \xi_{i+1}\}$$

if $2 \le i \le p-1$, to the open set

$$U_1 = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \theta(1-|z|) < \arg z < \arg \xi_2\}$$

if $i = 1$, and to the open set

$$U_p = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \arg \xi_p < \arg z < \theta(1-|z|)\}$$

if $i = p$, where $\eta(t) = (1-t)e^{i\theta(t)}$, $t \in [0,1]$, is a parametrization of the curve $C = \lim C_k$. Hence $\mathcal{P}_\epsilon$ is a Markov partition associated to f^* and compatible with f. It follows then from Lemma 6.6 that the transition functions $T_{I_i B_j}(f^*)^n$ are real analytic, proving that $\mathcal{P}_\epsilon$ satisfies (P1).

CLAIM A2. $\mathcal{P}_\epsilon$ satisfies property (P2) of Theorem 2.2.

We know that the partitions $\mathcal{P}_k$ have bounded distortion with a constant A that depends only on the open sets $U_{i,k}$ and $U_{i,k}^\gamma$, $2 \le i \le p-1$ (see Lemma 4.8). Since these open sets are converging to U_i and U_i^γ, respectively, we can choose the constant A independent of k. Hence $\mathcal{P}_\epsilon$ has bounded distortion, and therefore, by Lemma 6.8, satisfies (P2).

The properties (P3) and (P4) are satisfied by the construction of the partition $\mathcal{P}_\epsilon$.

Case B. f^* is analytic in an interval (ω_1, b) but not analytic in intervals of the form (b, ω_1).

We can suppose, w.l.o.g., that f^* is analytic and injective in $(\omega_1, \omega_1 + \epsilon)$ since otherwise $\Omega(R) \cap (\omega_1, \omega_1 + \epsilon) \ne \emptyset$, for some $0 < R < 1$, by Proposition 5.2. Then we would prove the properties (P1), (P2), (P3) and (P4) of Theorem 2.2 as in case A.

We have that $|(f')^*(x)| > a > 1$, $\forall x \in S^1$. Hence if f^* is analytic and injective in $(f^*)^j(\omega_1, \omega_1 + \epsilon)$, $0 \le j \le N$, then $\epsilon \cdot a^N \le 1$ and therefore $N \le B \log(1/\epsilon)$, where B is the constant $1/(\log a)$.

Let N_0 be the smaller value of j such that f^* is not analytic and injective in $(f^*)^j(\omega_1, \omega_1 + \epsilon)$. Then, it follows from Proposition 5.2 that there exist $\omega_0 \in \Omega(R)$ and $\alpha_0 \in (\omega_1, \omega_1 + \epsilon)$ with $(f^*)^{N_0}(\alpha_0) = \omega_0$, for some $0 < R < 1$. Write $\alpha_j = (f^*)^j(\alpha_0)$, $1 \le j \le N_0$.

Take a subset $\Xi(R) \subset \Omega(R)$ containing $\omega_0 = \xi_{i_0}$ and $\omega_1 = \xi_1$ and such that

$$\left(\#\Xi(R) + B \log \frac{1}{\epsilon}\right) \cdot \epsilon \le 1.$$

Let $\mathcal{P}_\epsilon$ be the partition whose intervals have extremities in the set $\Xi(R)$ and in $\{\alpha_j; 0 \le j \le N_0\}$.

We shall prove now that $\mathcal{P}_\epsilon$ satisfies the properties (P1) and (P2) of Theorem 2.2 assuming that $N_0 = 1$. If $N_0 > 1$, the proof is similar.

CLAIM B1. $\mathcal{P}_\epsilon$ satisfies property (P1) of Theorem 2.2.

Write $\mathcal{P}_\epsilon = \{J_0, J_1, \ldots, J_p\}$, where $J_0 = [\xi_1, \alpha_0]$, $J_2 = [\alpha_0, \xi_2]$ and $J_i = [\xi_i, \xi_{i+1}]$, if $2 \le i \le p$, and denote by $\alpha_{0,k}$ the unique zero of the equation $f_k(z) - \xi_{i_0,k}$ near α_0. Then $\mathcal{P}_\epsilon$ is the limit of the partitions $\mathcal{P}_k = \{J_{0,k}, J_{1,k}, \ldots, J_{p,k}\}$, where $J_{0,k} = [\xi_{1,k}, \alpha_{0,k}]$, $J_{1,k} = [\alpha_{0,k}, \xi_{2,k}]$ and $J_{i,k} = [\xi_{i,k}, \xi_{i+1,k}]$, if $2 \le i \le p$.

Since f^* is analytic at α_0, we can choose $0 < R < 1$, independent of k, such that there exist curves $C_{1,k}$ starting at $\xi_{i_0,k}$ and intersecting $\mathbf{D}_R$ whose liftings by f_k having $\alpha_{1,k}$ as base point, denoted by $L_{1,k}$, intersect $\mathbf{D}_R$. We can also choose the curves $C_{1,k}$ in such a way that they converge to a curve C_1.

By Proposition 4.7, $\mathcal{P}_k$ is a Markov partition with respect to f_k^* compatible with f_k. Moreover, the open sets $W_{i,k}$ associated with the intervals $J_{i,k}$ are converging to the open set

$$W_0 = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \theta(1 - |z|) < \arg z < \arg \alpha\}$$

if $i = 0$,

$$W_1 = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \arg \alpha < \arg z < \arg \xi_2\}$$

if $i = 1$,

$$W_{i_0-1} = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \arg \xi_{i_0-1} < \arg z < \theta_1(1 - |z|)\}$$

if $i = i_0 - 1$,

$$W_{i_0} = \{z \in \mathbf{C} \mid r < |z| < r^{-1}, \theta_1(1 - |z|) < \arg z < \arg \xi_{i_0+1}\}$$

if $i = i_0$, and $W_i = U_i$, for the other indexes, where $\eta_1(t) = (1 - t) \cdot e^{i\theta_1(t)}$, $t \in [0,1]$, is a parametrization of C_1. Hence $\mathcal{P}_\epsilon$ is a Markov partition with respect to f^* and compatible with f and therefore, by Lemma 6.6, $\mathcal{P}_\epsilon$ saitsifes (P1).

CLAIM B2. $\mathcal{P}_\epsilon$ satisfies property (P2) of Theorem 2.2.

Let Z be a neighborhood of $L_{1,k}$ where f_k has no critical values, $\forall k \geq k_0$. We can choose such a Z independent of k, by the analyticity of f^* at α_0. It follows then from Proposition 4.11 that the partitions $\mathcal{P}_k$ have bounded distortion, with a constant A independent of k. Hence $\mathcal{P}_\epsilon$ has bounded distortion and therefore, by Lemma 6.8, $\mathcal{P}_\epsilon$ satisfies (P2).

Properties (P3) and (P4) are satisfied by the construction of the partition $\mathcal{P}_\epsilon$.

We now have to analyse the possibility that f^* is analytic on an interval of the form (b, ω_1) and not analytic on any interval of the form (ω_1, b). But it is clear that this case is analogous to the case B. Similarly, the case where f^* is analytic in intervals of both forms, (b, ω_1) and (ω_1, b), also can be reduced to case B.

We shall now prove that for every sequence of ϵ's tending to zero, the corresponding sequence of partitions $\mathcal{P}_\epsilon$ satisfies properties (P5) and (P6) of Theorem 2.2.

(P5) is obviously satisfied. We must then prove (P6).

PROPOSITION 6.9.

$$\bigvee_{\substack{n\geq 0\\ l\geq 1}} (f^*)^{-n}\mathcal{P}_{\epsilon_l} = \mathcal{B}(\mathbf{S}^1).$$

PROOF. Write

$$A = \{x \in \mathbf{S}^1 \mid f^* \text{ is analytic and injective at } x\},$$

$$Q = \{q \in \mathbf{S}^1 \mid q \text{ is an extremity of an interval where } f^* \text{ is analytic}\}$$

and

$$B = (A \cup Q)^c.$$

Let T be a Borel set, $T \subset B$. It follows from Proposition 5.2 that, for each $x \in T$, there exists $l(x)$ such that $|\mathcal{P}_{\epsilon_{l(x)}}(x)| < \epsilon$, where $|E|$ denotes the diameter of E. Hence $T \in \bigvee_{l\geq 1}\mathcal{P}_{\epsilon_l}$.

Define $N: A \to \mathbf{N} \cup \{\infty\}$ by

$$N(x) = \max\{n \in \mathbf{N} \mid (f^*)^n(x) \in A\}.$$

CLAIM. N is measurable in $\bigvee_{n\geq 0, l\geq 1}(f^*)^{-n}\mathcal{P}_{\epsilon_l}$.

Indeed, observe that:

If $x \in N^{-1}(n)\setminus\bigcup_{q\geq 0}(f^*)^{-q}(Q)$ and $y \in N^{-1}(m)\setminus\bigcup_{q\geq 0}(f^*)^{-q}(Q)$, where $m > n$, then there exists l such that x and y are in different atoms of $(f^*)^{-(n+1)}\mathcal{P}_{\epsilon_l}$.

This observation is a consequence of Proposition 5.2 and proves the claim.

Let $R \subset A$ be a Borel set. To prove that $R \in \bigvee_{n\geq 0, l\geq 1}(f^*)^{-n}\mathcal{P}_{\epsilon_l}$ it is sufficient, by the claim above, to prove that $R_n = R \cap N^{-1}(n) \in \bigvee_{n\geq 0, l\geq 1}(f^*)^{-n}\mathcal{P}_{\epsilon_l}$, $\forall n \in \mathbf{N} \cup \{\infty\}$.

Case 1. $n = \infty$

In this case, if x and y are in the same atom of $(f^*)^{-n}\mathcal{P}_{\epsilon_l}$ then $|(x,y)| < a^{-n}$, where $a = \inf_{x\in \mathbf{S}^1}|(f')^*(x)|$. Hence $|R_\infty \cap (f^*)^{-n}\mathcal{P}_{\epsilon_l}(x)| < a^{-n}$, $\forall x \in R_\infty$. Therefore

$$R_\infty \in \bigvee_{\substack{n\geq 0\\ l\geq 1}} (f^*)^{-n}\mathcal{P}_{\epsilon_l}.$$

Case 2. $n < \infty$

In this case, if x and y are in the same atom of $(f^*)^{-(n+1)}\mathcal{P}_{\epsilon_l}$ then $(f^*)^{n+1}x$ and $(f^*)^{n+1}y$ are in the same atom of $\mathcal{P}_{\epsilon_l}$ and are in B, by the definition of N. Hence, if l is sufficiently large,

$$|R_n \cap (f^*)^{-(n+1)}\mathcal{P}_{\epsilon_l}(x)| < \epsilon.$$

Therefore

$$R_n \in \bigvee_{\substack{n\geq 0\\ l\geq 1}} (f^*)^{-n}\mathcal{P}_{\epsilon_l},$$

proving Proposition 6.9.

The proof of Theorem 2.2 is thus complete.

References

[A1] J. Aaronson, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. Henri Poincaré **14** (1978), 233–253.

[A2] J. Aaronson, *A remark on the exactness of inner functions*, J. London Math. Soc. (2)**23** (1981), 469–474.

[A-C] P. R. Ahern and D. N. Clark, *On inner functions with H^p-derivative*, Michigan Math. J. **21** (1974), 115–127.

[A-O-W] P. Arnoux, D. S. Ornstein and B. Weiss, *Cutting and stacking, interval exchanges and geometric models*, Isr. J. Math. **50** (1985), 160–168.

[Ca] C. Carathéodory, *Theory of Functions*, Vol. 2, Chelsea, New York, 1954.

[Co] P. Colwell, *Blaschke Products*, Univ. of Michigan Press, Ann Arbor, 1985.

[De] A. Denjoy, *Fonctions contractante le cercle* $|z| < 1$, C.R. Acad. Sci. Paris **182** (1926), 255–257.

[Du] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[F] J. L. Fernandez, *A note on entropy and inner functions*, Isr. J. Math. **53** (1986), 158–162.

[G] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[H] M. Heins, *On the finite angular derivatives of an analytic function mapping the open disk into itself*, J. London Math. Soc. (2)**15** (1977), 239–254.

[M] N. F. G. Martin, *On ergodic properties of restrictions of inner functions*, Ergodic Theory and Dynamical Systems **9** (1989), 137–151.

[N] J. H. Neuwirth, *Ergodicity of some mappings of the circle and the line*, Isr. J. Math. **31** (1978), 359–367.

[Pa] W. Parry, *Entropy and generators in ergodic theory*, Math. Lecture Notes Series, W. A. Benjamin, New York, 1969.

[Po1] Ch. Pommerenke, *On the iteration of analytic functions in a halfplane, I*, J. London Math. Soc. (2)**19** (1979), 439–447.

[Po2] Ch. Pommerenke, *On ergodic properties of inner functions*, Math. Ann. **256** (1981), 43–50.

[R] V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Am. Math. Soc., Transl., Ser. 2 **39** (1964), 1–36.

[W] J. L. Walsh, *Interpolation and analytic functions interior to the unit circle*, Trans. Am. Math. Soc. **34** (1932), 523–556.